Homework # 5 / Elasticity - Winter 2008
25/2/2008
Due: 5/3/2008
A complete and accurate solution to this homework will be partly averaged with the midterm.

Goal: To better understand the nature of the solution of a Boundary Value Problem (BVP) for an isotropic elastic body.

Note: Do all the steps in an index notation form. Consult the material in chapter 19 (pages 689-695) to solve the following problems.

Consider an isotropic elastic solid in region R under body forces $\mathbf{f}^0$, surface tractions $\mathbf{t}^0$ and/or surface displacements $\mathbf{u}^0$.

Problem 1.

(a) State all the boundary conditions and equations for the stress, strain and displacement that the solution to this BVP must satisfy.
(b) Restate the above equations for a displacement BVP and explain why the compatibility conditions are redundant in this case.
(c) Formulate a traction BVP from the equations of part (a). Explain why a unique solution to these equations is not guaranteed?
(d) Derive the Beltrami-Michell compatibility conditions. (i.e. derive equations 19.1.10a-b)

Problem 2.

(a) Under which conditions a displacement field is kinematically admissible? Compare them to the conditions from part (a) of problem 1.
(b) Call the set of all the kinematically admissible displacement fields for the above boundary-value problem, V^k. Now, consider the difference between two members of V^k. What is this displacement field called and is it a member of V^k? Why?
(c) Under which conditions a stress field is statically admissible? Compare them to the conditions from part (a) of problem 1.
(d) Call the set of all the statically admissible stress fields for the above boundary-value problem V^s. Now, consider the difference between two members of V^s. What is this stress field called and is it a member of V^s? Why? What boundary conditions does this stress field satisfy?
(e) Prove the reciprocity theorem for the isotropic elastic case: If $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are the solutions to the BVP under the states ($\mathbf{b}^{(1)}$, $\mathbf{t}^{(1)}$) and ($\mathbf{b}^{(2)}$, $\mathbf{t}^{(2)}$) acting on the same body respectively, then the following relation holds:

$$\int_R \rho b_i^{(1)} u_i^{(2)} dV + \oint_{\partial R} t_i^{(1)} u_i^{(2)} dR = \int_R \rho b_i^{(2)} u_i^{(1)} dV + \oint_{\partial R} t_i^{(2)} u_i^{(1)} dR \qquad (1)$$

(f) Write down the definition of potential energy. Take the variation of the potential energy function. Now, assume that the displacement in the potential energy function is the actual solution to the boundary-value problem. What would be the value of the variation of the potential energy in that case?

(g) Prove the theorem of minimum potential energy. (Start with the equation 19.3.6a and write down all the assumptions you make. Follow the steps given in page 695 to arrive at equation 19.3.7. Show all the details.)

Problem 3. (Extra Credit) Prove that the solution to a BVP is unique using the positive definiteness of the elasticity tensor.