General Solutions from Fundamental Solutions

"Reciprocal Theorem"

Consider

$$\left.\begin{aligned}\frac{\partial p}{\partial t}(\underset{\sim}{x},t) &= -\nabla\cdot\underset{\sim}{J}\{p\} + P(\underset{\sim}{x},t)\\ \frac{\partial q}{\partial t}(\underset{\sim}{x},t) &= -\nabla\cdot\underset{\sim}{J}\{q\} + Q(\underset{\sim}{x},t)\end{aligned}\right\}\quad(1)$$

where $\underset{\sim}{J}\{f\} = -D\nabla f$

Then

$$-\frac{\partial p}{\partial t'}(\underset{\sim}{x},t-t') \equiv \frac{\partial p}{\partial t}(\underset{\sim}{x},t-t')$$
$$= -\nabla\cdot\underset{\sim}{J}\{p(\underset{\sim}{x},t-t')\} + P(\underset{\sim}{x},t-t') \quad \ldots(2)$$

and (1) gives:

$$p(\underset{\sim}{x},t-t')\frac{\partial q}{\partial t'}(\underset{\sim}{x},t') + q(\underset{\sim}{x},t')\frac{\partial}{\partial t'}p(\underset{\sim}{x},t-t')$$

$$= q(\underset{\sim}{x},t')\nabla\cdot\underset{\sim}{J}\{p(\underset{\sim}{x},t-t')\} - p(\underset{\sim}{x},t-t')\nabla\cdot\underset{\sim}{J}\{q(\underset{\sim}{x},t')\}$$
$$+ p(\underset{\sim}{x},t-t')Q(\underset{\sim}{x},t') - q(\underset{\sim}{x},t')P(\underset{\sim}{x},t-t')$$

$$\equiv \nabla\cdot\left[q(\underset{\sim}{x},t')\underset{\sim}{J}\{p(\underset{\sim}{x},t-t')\} - p(\underset{\sim}{x},t-t')\underset{\sim}{J}\{q(\underset{\sim}{x},t')\}\right]$$
$$+ p(\underset{\sim}{x},t-t')Q(\underset{\sim}{x},t') - q(\underset{\sim}{x},t')P(\underset{\sim}{x},t-t') \quad \ldots(3)$$

Integration of (3) w.r.t. $\underline{x}, t'$ over $V \times (0,t)$ gives, with aid of Divergence theorem:

$$\int_V \left[q(\underline{x},t)\,p(\underline{x},0) - q(\underline{x},0)\,p(\underline{x},t) \right] dV(\underline{x})$$

$$= \int_A \int_0^t \left[q(\underline{x},t')\, J_\nu\{p(\underline{x},t-t')\} - p(\underline{x},t-t')\, J_\nu\{q(\underline{x},t')\} \right] dA\, dt'$$

$$+ \int_V \int_0^t \left[p(\underline{x},t-t')\, Q(\underline{x},t') - q(\underline{x},t')\, P(\underline{x},t-t') \right] dV\, dt' \qquad \ldots(4)$$

where $\quad J_\nu\{f(\underline{x},t)\} = \underline{J}\{f(\underline{x},t)\} \cdot \underline{\nu}$

$$= -D \frac{\partial f}{\partial \nu}(\underline{x},t) \equiv -D\, \underline{\nu} \cdot \nabla f(\underline{x},t)$$

$\underline{x}$ on A, is normal component of flux.

Construction of General Solution

Let $p(\underset{\sim}{x},t) = F(\underset{\sim}{x},t;\underset{\sim}{x}^*,0)$, $P(\underset{\sim}{x},t)=\delta(\underset{\sim}{x}-\underset{\sim}{x}^*)\delta(t)$

where
$$F(\underset{\sim}{x},t;\underset{\sim}{x}^*,t^*) \equiv F(\underset{\sim}{x},t-t^*;\underset{\sim}{x}^*,0) \equiv F(\underset{\sim}{x}^*,t;\underset{\sim}{x},t^*) \qquad (5)$$
is a fundamental solution, satisfying

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= -\nabla\cdot\underset{\sim}{J}\{F\} + \delta(\underset{\sim}{x}-\underset{\sim}{x}^*)\,\delta(t-t^*) \\ F(\underset{\sim}{x},t,\underset{\sim}{x}^*,t^*) &\equiv 0 \quad , \quad t \le t^* \end{aligned}\right\} \text{ in } \overset{*}{V} \qquad (6)$$

and an appropriate b.c. on $\partial V = A$. Then (4) becomes:

$$\begin{aligned} q(\underset{\sim}{x}^*,t) = &\int_V F(\underset{\sim}{x},t;\underset{\sim}{x}^*,0)\, q(\underset{\sim}{x},0)\, dV(\underset{\sim}{x}) \\ &+ \int_V\int_0^t F(\underset{\sim}{x},t;\underset{\sim}{x}^*,t')\, \underset{\sim}{Q}(\underset{\sim}{x},t')\, dV(\underset{\sim}{x})\,dt' \\ &+ \int_A\int_0^t \mu_\nu\{q,F\}\, dA(\underset{\sim}{x})\,dt' \end{aligned} \qquad (7)$$

where

$$\mu_\nu = q(\underset{\sim}{x},t')\, J_\nu\{F(\underset{\sim}{x},t;\underset{\sim}{x}^*,t')\} - F(\underset{\sim}{x},t;\underset{\sim}{x}^*,t')\, J_\nu\{q(\underset{\sim}{x},t')\}$$

Now we identify $g(\underset{\sim}{x}, t)$ with the solution $C(\underset{\sim}{x}, t)$ satisfying (1) and the b.c.

$$\alpha(\underset{\sim}{x})\, J_\nu\{C(\underset{\sim}{x}, t)\} + \beta(\underset{\sim}{x})\, C(\underset{\sim}{x}, t) = \gamma(\underset{\sim}{x}, t) \quad (8)$$

for $\underset{\sim}{x} \in \partial V$

whereas $F(\underset{\sim}{x}, t, \underset{\sim}{x}^*, t^*)$ satisfies the same condition with $\gamma(\underset{\sim}{x}, t) \equiv 0$. Then (7) can be written as

$$C(\underset{\sim}{x}, t) = \int_V F(\underset{\sim}{x}, t; \underset{\sim}{x}^*, 0)\, C^{(0)}(\underset{\sim}{x}^*)\, dV(\underset{\sim}{x}^*)$$

$$+ \int_V \int_0^t F(\underset{\sim}{x}, t; \underset{\sim}{x}^*; t')\, Q(\underset{\sim}{x}^*, t')\, dV(\underset{\sim}{x}^*)\, dt'$$

$$+ \int_A \int_0^t \mu_\nu \; dA(\underset{\sim}{x}^*)\, dt' \quad (9)$$

where

$$\mu_\nu = -\frac{\gamma(\underset{\sim}{x}^*)}{\alpha(\underset{\sim}{x}^*)} F(\underset{\sim}{x}^*, t; \underset{\sim}{x}, t') \overset{(\text{or})}{=} \frac{\gamma(\underset{\sim}{x}^*)}{\beta(\underset{\sim}{x}^*)} J_\nu^*\{F(\underset{\sim}{x}^*, t; \underset{\sim}{x}, t')\}$$

~~For~~ the standard b.c.'s either one or other is chosen from (10):

	α, β, γ	F	where
Dirichlet	$0, 1, C_0(\underset{\sim}{x}, t)$	G	.. C_0 is b.c. on C
Neumann	$1, 0, J_0(\underset{\sim}{x}, t)$	N	.. J_0 is b.c. on J_ν
Robin	$1, k, kC_\infty(\underset{\sim}{x}, t)$	R	.. k is transfer coeff.; C_∞ is external conc.