MAE 221 b
Solution MT3.17

a) The Green's function $G(t-t^*, z-z^*)$ satisfies

$$
\left.
\begin{aligned}
&\lambda \frac{\partial^2 G}{\partial t^2} + \frac{\partial G}{\partial t} = D \frac{\partial^2 G}{\partial z^2} + (1+\lambda\partial_t)\,\delta(t-t^*)\,\delta(z-z^*) \\
&t < t^* \qquad G \equiv 0\,, \\
&t > t^* \qquad \int_{-\infty}^{+\infty} G\,dz = \text{Const.}\,,
\end{aligned}
\right\} \quad (1)
$$

To derive it we may set $z^* = 0,\ t^* = 0$ and then replace t, z by $t-t^*$, $z-z^*$ in the result.
Applying the Laplace transformation: $\hat{G}(s,z) = \mathcal{L}\,G(t,z)$,

$$
\left.
\begin{aligned}
&(\lambda s^2 + s)\,\hat{G} = D\,\frac{d^2\hat{G}}{dz^2} + (1+s)\,\delta(z) \\
&\text{with} \quad (\lambda s^2+s)\int_{-\infty}^{\infty} \hat{G}\,dz = D\underbrace{\cancel{\int_{-\infty}^{+\infty}\frac{d^2\hat{G}}{dz^2}}}_{\to\,0} + 1 + \lambda s
\end{aligned}
\right\} \quad (2)
$$

This is satisfied by

$$
\hat{G}(s,z) = \frac{(\lambda s+1)\, e^{-\sqrt{\frac{\lambda s^2+s}{D}}\,|z|}}{2D\sqrt{\frac{\lambda s^2+s}{D}}}
$$

From the result

$$
\mathcal{L}^{-1}\,\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} = e^{-\frac{1}{2}at}\, I_0\!\left(\frac{a}{2}\sqrt{t^2-k^2}\right) H(t-k), \quad \text{for } k \geq 0
$$

(See Abramowitz & Stegun, Handbook of Math. Funct., p.1027) where I_0 is the modified Bessel function of the first kind and H is the Heaviside step function, it follows, on taking $k = \sqrt{\frac{\lambda}{D}}|z|$, $a = \frac{\cancel{D}\,1}{\lambda}$, that

$$(\lambda\partial_t + 1)$$

$$G(t,z) = \mathcal{L}^{-1}\hat{G}(s,z) = \frac{1}{2\sqrt{D\lambda}}\,\lambda\, e^{-\frac{\cancel{D}^{1}t}{2\lambda}} I_0\left\{\frac{\cancel{D}^{1}}{2\lambda}\sqrt{t^2-\frac{\lambda z^2}{D}}\right\} H\left(t-\sqrt{\frac{\lambda}{D}}|z|\right) \quad \ldots (3)$$

b) From the asymptotic expansion

$$I_0(x) \sim \frac{e^{x}}{\sqrt{2\pi x}}\left\{1+O(1/x)\right\}$$

(Same reference as above, p. 377), it follows that

$$G(t,z) \sim \frac{1}{2\sqrt{D\lambda}}\,\frac{e^{-\frac{Dt}{2\lambda}+\frac{D}{2\lambda}\sqrt{t^2-\frac{\lambda z^2}{D}}}}{\sqrt{\frac{2\pi D}{2\lambda}\sqrt{t^2-\frac{\lambda z^2}{D}}}} \sim \frac{e^{-\frac{z^2}{4Dt}}}{2\sqrt{\pi D t}}$$

for $\lambda \to 0$.

Otherwise, (3) has a cut-off at $|z| = \sqrt{\frac{D}{\lambda}}\,t$ or $|z| = u\,t$, where $u = \sqrt{\frac{D}{\lambda}}$ is a signal speed.

c) $$G(t,x,y,z) = G(t,x)\,G(t,y)\,G(t,z),$$

by "separation" of variables.