Solution to JDG MT 10.4

Let x be dimensionless distance. Eq'ns become

$$\frac{\mathcal{D}_A}{L^2}\frac{d^2C_A}{dx^2} \underset{①}{=} k\left(C_A - \frac{1}{K}C_B\right) \underset{②}{=} -\frac{\mathcal{D}_B}{L^2}\frac{d^2C_B}{dX^2}$$

$$@\ x=0\ \ C_A=\bar{C},\ \frac{dC_B}{dx}=0,\ @\ x=1\ \ C_A=\underline{C},\ \frac{dC_B}{dX}=0 \qquad ③$$

Assume solutions of form:

$$C_A = a\,e^{mx}, \quad C_B = b\,e^{mx}$$

giving, by ① and ②,

$$m^2(m^2-\lambda^2) = 0, \text{ and } b = K\left(1-\frac{m^2}{\gamma_A}\right)a, \quad \ldots ④$$

with $m = \pm\lambda$, $m = 0, 0$ as roots

where

$$\lambda^2 = \frac{kL^2}{\mathcal{D}_A} + \frac{kL^2}{K\mathcal{D}_B} = \gamma_A + \gamma_B \qquad ⑤$$

$$\gamma_A = \frac{kL^2}{\mathcal{D}_A}, \quad \gamma_B = \frac{kL^2}{K\mathcal{D}_B}$$

So

$$C_A = \left[a_1 e^{\lambda x} + a_2 e^{-\lambda x}\right] + \left[a_3 x + a_4\right] \qquad ⑥$$

$$C_B = \underbrace{K\left(1-\frac{\lambda^2}{\gamma_A}\right)\left[a_1 e^{\lambda x} + a_2 e^{-\lambda x}\right]}_{m=\pm\lambda} + \underbrace{K\left[a_3 x + a_4\right]}_{m=0}$$

Applying B.C.'s ③ gives, after some algebra,

$$C_A = C^* + \frac{\Delta C}{2\alpha}\left\{\frac{\sinh\lambda\left(\frac{1}{2}-x\right)}{\sinh\lambda/2} + \left[\left(1-\frac{\lambda^2}{\gamma_A}\right)\lambda\coth\frac{\lambda}{2}\right]\left(x-\frac{1}{2}\right)\right\}$$

$$C_B = KC^* + K\left(1-\frac{\lambda^2}{\gamma_A}\right)\frac{\Delta C}{2\alpha}\left\{\frac{\sinh\lambda\left(\frac{1}{2}-x\right)}{\sinh\lambda/2} + \lambda\coth\frac{\lambda}{2}\left(x-\frac{1}{2}\right)\right\} \qquad \ldots ⑦$$

(cont'd)

Sol'n. JDG MT 4.

where $c^* = \dfrac{\bar{c}+\underline{c}}{2}$, $\Delta C = \bar{c} - \underline{c}$,

$$\alpha = 1 - \left(1 - \frac{\lambda^2}{\gamma_A}\right)\frac{\lambda}{2}\coth\frac{\lambda}{2}$$

Then,

$$\phi = \frac{-\mathcal{D}_A\left(\frac{dC_A}{dx}\right)_{x=0}}{\mathcal{D}_A(\bar{c}-\underline{c})} = \frac{\lambda^3}{2\gamma_A}\,\frac{\coth\lambda/2}{1+\left(\frac{\lambda^2}{\gamma_A}-1\right)\frac{\lambda}{2}\coth\frac{\lambda}{2}} \qquad (8)$$

Note that for $\lambda \to \infty$

$$\phi \to \phi_{eq} = \frac{\lambda^2/\gamma_A}{\lambda^2/\gamma_A - 1} = \frac{\gamma_A + \gamma_B}{\gamma_B} = 1 + \frac{K\mathcal{D}_B}{\mathcal{D}_A}$$

So that $\lambda^2/\gamma_A = \phi_{eq} / (\phi_{eq} - 1)$

and (8) can be rewritten as

$$\phi = \frac{\phi_{eq}\, w}{\phi_{eq} - 1 + w} \qquad (9)$$

$$w = \frac{\lambda}{2}\coth\frac{\lambda}{2}, \quad \lambda = \sqrt{\frac{\gamma_A\, \phi_{eq}}{\phi_{eq} - 1}}$$

(Q.E.D)