MAE105
First Midterm Exam
(open book; closed notes; no computers, no calculators, no cell phones)

Name:_____

Time: 3:50 to 4:50pm

Date: April 17, 2008

Problem 1:

Consider the following diffusion PDE:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad t > 0, \qquad 0 < x < \pi, \tag{1}$$

with the following boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad u(\pi, t) = 0. \tag{2}$$

(a) (0.5 Point) Based on the method of "separation of variables", set $u(x,t) = \phi(x)\, G(t)$, differentiate as necessary, and substitute into the PDE (1).

(b) (0.5 Point) Collect terms such that a function of only x is set equal to a function of only t, and use this fact to find two ODE's, one for $\phi(x)$, and the other for $G(t)$, in such a way that the solution for $\phi(x)$ would be periodic.

(c) (0.5 Point) Write down the general solution of the ODE for $\phi(x)$. Is your solution periodic?

(d) (0.5 Point) Write down the general solution of the ODE for $G(t)$. Does your solution decay exponentially in time?

(e) (1 Point) Use the boundary conditions (2) to obtain the values of $\phi'(x)$ at $x = 0$ and $\phi(x)$ at $x = \pi$.

(f) (2 Points) Use the values of $\phi'(x)$ at $x = 0$ and $\phi(x)$ at $x = \pi$ to find the eigenvalues and the corresponding eigenfunctions of this problem. Show by direct calculation that the eigenfunctions are in fact orthogonal. Is zero an eigenvalue in the present case?

(g) (2 Points) Consider the following initial condition:

$$u(x, 0) = f(x) = 13\cos\left(\frac{x}{2}\right) + 5\cos\left(\frac{3x}{2}\right), \qquad 0 < x < \pi. \tag{3}$$

(h) (1 Point) Based on the above results, write down the series solution (with explicit coefficients) of the initial-boundary-value problem defined by expressions (1), (2), and (3), *i.e.,* write down the series solution for $u(x, t)$.

Problem 2:

Consider the following differential equation:

$$\frac{d^2u}{dx^2} - x^2 = 0, \qquad 0 < x < 2, \tag{4}$$

with the following boundary conditions:

$$u(0) = 1, \qquad u(2) = 4. \tag{5}$$

(a) (1 Point) Integrate the equation twice to obtain an explicit expression for $u(x)$ which must include two integration constants.

(b) (1 Point) Use the boundary conditions to find the constants of integration and write down explicitly the final solution.

Now consider the following first-order DE:

$$\frac{du}{dx} - x\cos x = 0, \qquad 0 < x, \qquad u(0) = 5. \tag{6}$$

(c) (2 Points) Integrate the equation to obtain an explicit expression for $u(x)$ which must include one integration constant; use integration by parts.

(d) (1 Point) Use the initial condition to find the constant of integration and write down explicitly the final solution.

Make sure to write your name at the top of each page.

Good luck!