May 06, 2008

MAE 105 Homework #7
Due: Tuesday, 05/13/08

PROBLEM 1:

Consider the diffusion equation in a rectangular region,

$$\frac{\partial u(x, y, t)}{\partial t} - \nabla^2 u(x, y, t) = 0, \qquad 0 < x < L, \qquad 0 < y < H, \qquad t > 0, \tag{1}$$

with boundary conditions

$$u(0, y, t) = 0, \qquad u(L, y, t) + \alpha \frac{\partial u(L, y, t)}{\partial x} = 0, \qquad u(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0,$$

where α is a constant, and the initial condition,

$$u(x, y, 0) = \beta(x, y).$$

(a) (0.5 Point) Set $u(x, y, t) = h(t)\,\phi(x, y)$. Find the ODE for $h(t)$ and the PDE for $\phi(x, y)$.

(b) (0.5 Point) Find the general solution for $h(t)$ which decays in time.

(c) (1 Point) Set $\phi(x, y) = f(x)\,g(y)$. Find the ODE's for $f(x)$ and $g(y)$, and the boundary conditions for each of these functions.

(d) (0.5 Point) For part (d) only, set $\alpha = -1$. Sketch how to find the eigenvalues for $f(x)$ graphically.

For the remaining part of this PROBLEM (parts (e) to (i)), set $\alpha = 0$.

(e) (0.5 Point) Apply the B.C.'s and find the general solution fo $g(y)$.

(f) (0.5 Point) Apply the B.C.'s and find the general solution fo $f(x)$.

(g) (0.5 Point) Find the eigenvalues and eigenfunctions associated with $\phi(x, y)$.

(h) (0.5 Point) Write out the infinite series solution for $u(x, y, t)$.

(i) (0.5 Point) Use the following expression for the initial condition to find the constants in the infinite series solution:

$$\beta(x, y) = 6 \sin(2\,\pi\, x/L)\sin(1.5\,\pi\, y/H).$$

PROBLEM 2 (2 Points):

Consider the following ODE:

$$\frac{d^2\phi}{dx^2} + (2x - \sin x)\frac{d\phi}{dx} + [\lambda\,\sigma(x) + \gamma(x)]\,\phi = 0, \tag{2}$$

where $\sigma(x) > 0$ and $\gamma(x)$ are given functions. Multiply through by the yet unknown function $p(x)$, and then find an expression for $p(x)$ such that (2) become a Sturm-Liouville DE.

PROBLEM 3 (2 Points):

Consider the following ODE:

$$\frac{d}{dx}[(\sin x + 1)\,\frac{d\phi}{dx}] + \lambda\,\phi = 0\,, \qquad 0 < x < \pi\,,$$

with boundary conditions

$$\phi(0) = \phi(\pi) = 0\,.$$

Use the Rayleigh quotient with $\phi_t = \sin x$ to estimate the first eigenvalue of this problem.

Note 1: To receive full credit, *all steps must be neatly shown.* Writing down the final results will receive no credit.

Note 2: Homeworks must be turned in at the start of due-date class. Late homeworks will be graded but *will receive zero credit*.