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Brief paper

Subspace identification with eigenvalue constraints[☆]

Daniel N. Miller [1], Raymond A. de Callafon

Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, USA

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ABSTRACT

Standard subspace methods for the identification of discrete-time, linear, time-invariant systems are transformed into generalized convex optimization problems in which the poles of the system estimate are constrained to lie within user-defined convex regions of the complex plane. The transformation is done by restating subspace methods such as the minimization of a Frobenius norm affine in the estimate parameters, allowing the minimization to be augmented with convex constraints. The constraints are created using linear-matrix-inequality regions, which generalize standard Lyapunov stability to arbitrary convex regions of the complex plane. The algorithm is developed for subspace methods based on estimates of the extended observability matrix and methods based on estimates of state sequences, but it is extendable to all subspace methods. Simulation examples demonstrate the utility of the proposed method.

1. Introduction

When identifying models of systems from measured data, it is often desirable that the identified model behave in agreement with prior knowledge of the system. This is ordinarily limited to basic knowledge of system stability or an assumed model order, but other times this knowledge is derived from first-principles laws that govern the underlying system dynamics. The system identification literature, however, tends to focus on “black-box” modeling approaches that limit the type of constraints that may be incorporated into the identification process.

One possible reason for the lack of constrained identification procedures is that researchers in the field of system identification are most frequently interested in constructing models for control-system design, and the idea of artificially constraining the dynamic behavior of an identified model to match *a priori* assumptions appears counterproductive for this purpose. Many practitioners outside of control design, however, are constrained to using pre-parameterized models for reasons unrelated to control. Faced with a lack of robust tools to identify constrained models, these practitioners often resort to “white-box” approaches which describe the desired model as a mass–spring–damper or resistor–capacitor–inductor network and then identify the parameters of each network element.

Such “white-box” approaches often excessively limit the acceptable model set, since only a small number of parameters may be identified this way before variance issues result in models with undesirable parameters, such as negative mass. These models identified with “white-box” methods are frequently provided to practicing controls engineers with little or no disclosure of their origin, so the development of constrained identification procedures should benefit the field of control systems as a whole.

An example of how constrained identification methods may be used to satisfy *a priori* parameterizations has been recently published by the authors and industry collaborators in Miller, Hulett, Mclaughlin, and de Callafon (in press). A constrained step-based-realization procedure, originally developed by the authors in Miller and de Callafon (2012), was used to identify the transient thermodynamic response of power electronics to construct models that matched an existing industry specification. The method provides an alternative to the industry-standard method of identifying the elements of a representational resister–capacitor system by fitting a curve to a numerically differentiated step response, which has difficulties with noisy data. The method in Miller and de Callafon (2012) can be considered a special case of the method presented in this paper.

Another possible reason for the lack of constrained identification procedures is that the classical prediction-error framework relies on the optimization of possibly non-convex cost functions. Such optimizations are already computationally challenging without adding possibly non-convex constraints. Subspace identification methods, by contrast, use a fixed number of linear algebra

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E-mail addresses: dnmiller@gmail.com (D.N. Miller), callafon@ucsd.edu (R.A. de Callafon).

[1] Tel.: +1 858 356 7003; fax: +1 858 822 3107.

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operations to achieve consistent estimates even in the presence of colored noise. While generally non-optimal in a prediction-error or maximum-likelihood sense, all subspace methods nonetheless minimize some Frobenius norm, the argument of which is affine in the identification parameters. The inherent convexity of the minimization has so far been largely, though not entirely, unexploited.

Examples of subspace identification methods which incorporate convex-optimization techniques include Lacy and Bernstein (2003), in which a linear-matrix inequality (LMI) framework is proposed to constrain the eigenvalues of system estimates to be stable, and Hoagg, Lacy, Erwin, and Bernstein (2004a) and McKelvey and Moheimani (2005), which use similar LMI frameworks to restrict estimates to positive real systems. In Hoagg, Lacy, Erwin, and Bernstein (2004b), this framework is extended to provide a lower bound on the natural frequencies of the poles of the identified model, creating a convex optimization procedure which restricts the eigenvalues to a *non*-convex region of the convex plane; the parameterization used, however eliminates the possibility of also restricting the eigenvalues to lie within convex regions of the complex plane, such as the unit circle.

A number of methods which enforce stability for subspace estimates without convex optimization techniques have been proposed. In Van Gestel, Suykens, Van Dooren, and De Moor (2001), a regularization-based method was proposed to iteratively adjust an initial estimate until its spectral radius reached a given bound. This was also shown to be equivalent in a special case to the data-augmentation approach of Chui and Maciejowski (1996), in which block rows are appended to the estimate of the extended observability matrix to ensure the stability of a least-squares estimate from its row space. In Goethals, Van Gestel, Suykens, Van Dooren, and De Moor (2003), a regularization-based approach is used to guarantee that the resulting estimate of the stochastic subsystem is positive real.

Additional subspace methods that incorporate prior knowledge of the system include Okada and Sugie (1996), which develops a method for the case in which some of the pole locations of the system are known beforehand. In Trnka and Havlena (2009), constraints were developed to fix the steady-state gain of the system and minimize a form of numerical derivative of the system step response. In Alenany, Shang, Soliman, and Ziedan (2011), an equality-constrained quadratic program was developed to enforce a lower-block-triangular structure of a matrix of Markov parameters, guaranteeing causality of the system, and to constrain the steady-state gain.

In this paper, we propose a new framework to impose general eigenvalue constraints for subspace identification problems. The eigenvalue constraints are constructed using the concept of LMI regions (Chilali & Gahinet, 1996), which generalize standard Lyapunov stability to convex regions of the complex plane. The generality of our method allows for the eigenvalues of the estimate to be constrained to any convex region of the complex plane that can be expressed as the intersection of ellipsoids, parabolas, or half-spaces symmetric about the real axis.

Our approach generalizes the methods proposed in Hoagg et al. (2004a) and Lacy and Bernstein (2003), which over-constrain the discrete-time Lyapunov inequalities. We also present a stability criteria which over-constrains the discrete-time Lyapunov inequalities as a special case of the general method, but our constraint has an exact geometric interpretation in the complex plane. In addition to a stability constraint, we also provide constraints that require eigenvalues to have positive real parts and/or zero imaginary parts.

Formulas are developed for incorporating eigenvalue constraints into two popular general methods of subspace identification: identification from an estimate of the extended observability matrix, and identification from an estimate of a sequence of states. Though constraints for only two methods are proposed, the methods presented in this paper are straightforward to extend to any method in which the estimate $\hat{A}$ can be formulated as the solution to a Frobenius-norm minimization. This includes standard subspace methods such as the ones found in Katayama (2005) and Verhaegen and Verdult (2007) as well as realization-based methods such as the Eigensystem Realization Algorithm/OKID (Juang, 1997) and methods which use dynamic invariance of the output (Miller & de Callafon, 2010).

The rest of this paper is outlined as follows: In Section 2, some basic notation and Frobenius-norm interpretations for subspace identification are introduced. Formulas for subspace identification methods that identify estimates from both the extended observability matrix and a sequence of states are derived. In Section 3, Linear Matrix Inequality regions are introduced and some useful constraints for identification are discusses and formulated. Section 4 incorporates the constraints into the identification problem. Section 5 presents some numerical examples, and Section 6 concludes the discussion.

2. Subspace identification

We consider the identification of the parameters of a linear, time-invariant, discrete-time system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

in which $u(t) \in \mathbb{R}^{n_u}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{n_y\times n}$, and $D \in \mathbb{R}^{n_y}$.

While deriving the methods in this paper, we will assume that either an estimate of the extended controllability matrix or an estimate of a state sequence has been made in an unbiased manner from adequately exciting data.

The standard notation of indicating positive definiteness and semi-definiteness of a matrix M as $M > 0$ and $M \geq 0$, respectively, is used; $\|\cdot\|_F$ denotes the Frobenius matrix norm; $(\cdot)^\dagger$ denotes the pseudoinverse, left or right when appropriate.

2.1. Identification from an estimate of the extended observability matrix

The first type of subspace identification methods considered are those that generate estimates of A from a shift-invariant property of the extended observability matrix, typically referred to as MOESP-type methods (Verhaegen & Verdult, 2007). Given an estimate the extended observability matrix of a system

$$\mathcal{O}^T = \begin{bmatrix} C^T & (CA)^T & \cdots & (CA^k)^T \end{bmatrix},$$

with respect to an arbitrary state basis, define

$$O_0 = \mathcal{O}_{(1:n_y(k-1),\ :)} \quad \text{and} \quad O_1 = \mathcal{O}_{(n_y+1:n_yk,\ :)} \tag{2}$$

in which the subscripts of $\mathcal{O}$ denote MATLAB-style indexing. Since $O_0A = O_1$, we may compute an estimate $\hat{A}$ from an estimate $\hat{\mathcal{O}}$ by minimizing the cost function

$$J^0_{\mathcal{O}}(A) = \left\| \hat{\mathcal{O}}_0 A - \hat{\mathcal{O}}_1 \right\|_F, \tag{3}$$

which is convex in A. In the unconstrained case, this is a linear least-squares problem with the analytic minimum

$$\hat{A} = \hat{\mathcal{O}}_0^\dagger \hat{\mathcal{O}}_1. \tag{4}$$

2.2. Identification from an estimate of a sequence of states

Alternatively, an estimate of A may be found from an estimated state sequence. Given a state sequence

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(j) \end{bmatrix},$$

define

$$X_0 = X_{(:,\ 1:j)}, \qquad \text{and} \qquad X_1 = X_{(:,\ 2:j+1)},$$

in which the subscripts again denote MATLAB-style indexing. Also define an input sequence

$$U = \begin{bmatrix} u(0) & u(1) & \cdots & u(j-1) \end{bmatrix}.$$

Since

$$X_1 = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U \end{bmatrix},$$

we may compute estimates $\hat{A}$ and $\hat{B}$ from an estimate $\hat{X}$ by minimizing the cost function

$$J_X^0(A,\ B) = \left\| \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \hat{X}_0 \\ U \end{bmatrix} - \hat{X}_1 \right\|_F, \tag{5}$$

which is also convex in $[A\, B]$ and in the unconstrained case has the analytic minimum

$$\begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} = \hat{X}_1 \left(\begin{bmatrix} \hat{X}_0 \\ U \end{bmatrix} \right)^{\dagger}. \tag{6}$$

3. Formulating eigenvalue constraints

To develop eigenvalue constraints, we use the concept of LMI regions, first introduced in Chilali and Gahinet (1996), which define convex regions of the complex plane as LMIs. We then provide some example regions that are useful for identification purposes.

3.1. LMI regions

An LMI region is a convex region $\mathcal{D}$ of the complex plane, defined in terms of a symmetric matrix α and a square matrix β, as

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) \geq 0\} \tag{7}$$

where

$$f_{\mathcal{D}}(z) = \alpha + \beta z + \beta^T \bar{z}. \tag{8}$$

LMI regions generalize standard notions of stability for continuous and discrete time systems, and the function parameters α and β may be used to form Lyapunov-type inequalities.

We repeat the central theorem of Chilali and Gahinet (1996) here for future reference.

Theorem 1. *The eigenvalues of a matrix A lie within an LMI region given by (7) if and only if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P = P^T > 0, \qquad \mathcal{M}_{\mathcal{D}}(A,\ P) \geq 0 \tag{9}$$

in which

$$\mathcal{M}_{\mathcal{D}}(A,\ P) = \alpha \otimes P + \beta \otimes (AP) + \beta^T \otimes (AP)^T. \tag{10}$$

The original definition of an LMI region in Chilali and Gahinet (1996) has $<$ in place of $\geq$ for (7) and (9). We adopt the above definition instead so that our results are straightforward to implement as a semi-definite program and because the real and imaginary axes cannot be parameterized as LMI regions if (7) uses a strict inequality. This change does not affect the proofs of Chilali and Gahinet (1996), since they are based on the relationship

$$\left(I \otimes v^H\right) \mathcal{M}_{\mathcal{D}}(A,\ P)\, (I \otimes v) = \left(v^H P v\right) f_{\mathcal{D}}(\lambda)$$

where λ is an eigenvalue and v is the corresponding left eigenvector of A. Because P is positive definite, the signs of $\mathcal{M}_{\mathcal{D}}$ and $f_{\mathcal{D}}$ need only to be equal, not necessarily negative, as they are in Chilali and Gahinet (1996).

The intersection of two LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ is also an LMI region, described by the matrix function

$$f_{\mathcal{D}_1 \cap \mathcal{D}_2}(z) = \begin{bmatrix} f_{\mathcal{D}_1}(z) & 0 \\ 0 & f_{\mathcal{D}_2}(z) \end{bmatrix}. \tag{11}$$

Note that in general the $(\alpha,\ \beta)$ pair that describes an LMI region is not unique.

3.2. Some LMI regions useful for identification

In the following we derive some LMI regions useful for identification purposes. Of course the user need not be limited by these; LMI regions can be constructed for any convex intersection of half-spaces, ellipsoids, and parabolas symmetric about the real axis.

3.2.1. Discrete-time stable eigenvalues

Stable system estimates are often desirable in the identification problem. Standard subspace methods, however, do not guarantee stability of the identified model. To provide some known degree of stability for the identified model, we may constrain eigenvalues to the disc of radius $1 - \delta_s$.

Fact 1. *The set*

$$\mathcal{S} = \{z \in \mathbb{C} : |z| \leq 1 - \delta_s,\ 0 \leq \delta_s \leq 1\}$$

is equivalent to the LMI region $f_{\mathcal{S}}(z) \geq 0$,

$$f_{\mathcal{S}}(z) = (1 - \delta_s) I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{z}. \tag{12}$$

This region applied with Theorem 1 results in the discrete-time Lyapunov stability condition if $\delta_s = 0$. It is also similar, though not identical, to the LMI constraint proposed in Lacy and Bernstein (2003). In (12), however, the relaxation parameter δ_s has a specific interpretation in the complex plane.

3.2.2. Eigenvalues with positive real parts

Discrete-time systems with negative real poles cannot be transformed into continuous systems that generate real-valued signals without increasing the model order (Kollár, Franklin, & Pintelon, 1996). Thus if the intention is to identify a continuous-time model of a pre-specified order, it is generally desirable to restrict the eigenvalues of the identified discrete-time model to have positive real parts. Consequently, we wish to construct an LMI region that describes the positive right-half plane. This region should also be parameterized so that the region begins some distance away from the imaginary axis.

Fact 2. *The set*

$$\mathcal{P} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq \delta_p,\ \delta_p \geq 0\}$$

is equivalent to the LMI region $f_{\mathcal{P}}(z) \geq 0$,

$$f_{\mathcal{P}}(z) = \delta_p \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{z}. \tag{13}$$

3.2.3. Eigenvalues with zero imaginary parts

Many thermodynamic processes are known to have strictly real eigenvalues. A simple example is the warming and cooling of the ambient air temperature in a room. Both intuition and the laws of thermodynamics tell us that the air temperature of the room cannot exceed the temperature of any heat source. However, a model identified from noisy data may have eigenvalues with nonzero imaginary parts, effectively allowing the air accumulate heat potential and to overshoot the temperature of its heat sources, which is impossible in living conditions. Thus we would like to construct an LMI region that describes the real number line.

Fact 3. *The real number line $\mathbb{R}$ is equivalent to the LMI region $f_{\mathbb{R}}(z) \geq 0$,*

$$f_{\mathbb{R}}(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{z}.$$

This constraint, however, is computationally unfriendly for many numerical optimization procedures, since it is effectively using two inequalities to define an equality, which can create problems for interior-point-based solvers. Instead, we include a parameter to describe an arbitrarily small band around the real axis in the complex plane.

Fact 4. *The set*

$$\mathcal{R} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \delta_r,\ \delta_r \geq 0\}$$

is equivalent to the LMI region $f_{\mathcal{R}}(z) \geq 0$,

$$f_{\mathcal{R}}(z) = 2\delta_r I_2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{z}. \tag{14}$$

4. Subspace identification with eigenvalue constraints

We now show how constraints based on LMI regions may be incorporated into the subspace identification problem. We first incorporate the constraints into methods based on the extended observability matrix, and then into methods based on estimated state sequences.

4.1. Extended observability matrix approach

The goal is to formulate a convex cost function similar to (3) that can be solved subject to constraints of the form (9). Though (3) is convex in A, the constraint (10) contains the product AP. We therefore modify (3) to also contain the product AP by adding P as a right-hand weighting matrix to (3), forming the cost function

$$J_{\mathcal{O}}^1(A,\ P) = \left\| \hat{\mathcal{O}}_0 AP - \hat{\mathcal{O}}_1 P \right\|_F. \tag{15}$$

Note that the unconstrained minimizer of (15) is still given by (4), regardless of the value of P. This is not necessarily true, however, when A is constrained to be within an arbitrary convex set. This can be seen from the gradient of $J_{\mathcal{O}}^1(A,\ P)$ with respect to $\mathrm{vec}(A)$,

$$\frac{\partial J_{\mathcal{O}}^1(A,\ P)}{\partial \mathrm{vec}(A)} = \frac{1}{2} \frac{1}{J_{\mathcal{O}}^1(A,\ P)} \mathrm{vec}\left[\hat{\mathcal{O}}_0^T \left(\hat{\mathcal{O}}_0 A - \hat{\mathcal{O}}_1 \right) P^2 \right]^T.$$

This is zero if $A = \hat{\mathcal{O}}_0^{\dagger} \hat{\mathcal{O}}_1$; otherwise, it depends on P. Thus (15) is only guaranteed to have the same optimality conditions as (3) at every point in a set if $P = I$, which is not guaranteed to satisfy the constraint (9). The effects of this change in optimality conditions were observed in the simulation examples in Lacy and Bernstein (2003), though the cause was left unexplained.

We still must re-parameterize (15) to be affine in the parameters in order to formulate the constrained optimization as a convex optimization. Letting $Q = AP$, we form the following convex optimization problem with convex constraints:

Given an estimate $\hat{\mathcal{O}}$ and an LMI region described by parameters α and β,

$$\begin{aligned} &\text{minimize} && J_{\mathcal{O}}^c(Q,\ P) \\ &\text{subject to} && \mathcal{M}(Q,\ P) \geq 0, \\ &&& P = P^T > 0 \end{aligned} \tag{16}$$

in which

$$J_{\mathcal{O}}^c(Q,\ P) = \left\| \hat{\mathcal{O}}_0 Q - \hat{\mathcal{O}}_1 P \right\|_F,$$
$$\mathcal{M}(Q,\ P) = \alpha \otimes P + \beta \otimes Q + \beta^T \otimes Q^T,$$

and the subscripts of $\hat{\mathcal{O}}$ are given by (2).

Once Q and P are solved for, we let $\hat{A} = QP^{-1}$. This solution, however, allows for arbitrarily small Q and P, which may introduce errors in the computation of $\hat{A}$. To improve numerical conditioning of the problem, we add the constraint

$$\mathrm{trace}(P) = C \tag{17}$$

where C is some constant. Although any C is valid, we recommend choosing $C = n$ to allow for the possible solution $P = I$.

At this point we should remark that although the global minimizer (4) might be in the set of feasible points, numerical optimization tools may not be able to find it exactly. Optimization routines based on primal–dual gap methods (Boyd & Vandenberghe, 2004) may deviate from (4) even when it is feasible and supplied as an initial value. This is because, although the analytic solution to primal and dual problems is the same, the numerical solution might not be. Such numerical difficulties become more common when the row dimension of $\hat{\mathcal{O}}$ becomes very large. In practice, it is best to confirm that the eigenvalues of (4) do not satisfy the LMI region's characteristic equation before solving the convex optimization problem.

4.2. State sequence approach

Similar to the approach used in the extended observability case, we wish to modify (5) so that it contains the product AP instead of A alone. We again take the approach of modifying the expression inside the Frobenius, this time with a right-hand weighting

$$R = \begin{bmatrix} \hat{X}_0 \\ U \end{bmatrix}^{\dagger} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}.$$

The right-hand weighted cost is then

$$\begin{aligned} J_X^1(A,\ B) &= \left\| \left(\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U \end{bmatrix} R - \hat{X}_1 \right) \right\|_F \\ &= \left\| \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A^* & B^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \right\|_F \end{aligned}$$

where

$$\begin{bmatrix} A^* & B^* \end{bmatrix} = \hat{X}_1 \left(\begin{bmatrix} \hat{X}_0 \\ U \end{bmatrix} \right)^{\dagger}$$

is the solution to the unconstrained minimization problem. Since the constraints do not depend on the parameter B, we let $B = B^*$, resulting in the constrained cost function

$$J_X^2(A) = \left\| AP - A^* P \right\|_F. \tag{18}$$

Hence in the state-sequence case the optimization is equivalent to finding the closest estimate in the Frobenius norm sense that satisfies the constraints. In this case, the gradient of (18) is

$$\frac{\partial J_X^2(A)}{\partial \mathrm{vec}(A)} = \frac{1}{J_X^2(A)} \mathrm{vec}\left[(A - A^*)^T P^2 \right]^T,$$

which again has dependence on P that may prevent the gradients of the weighted and unweighted costs from being equal.

To form an affine parameterization of (18), we again let $Q = AP$, and provide the following convex optimization problem with convex constraints:

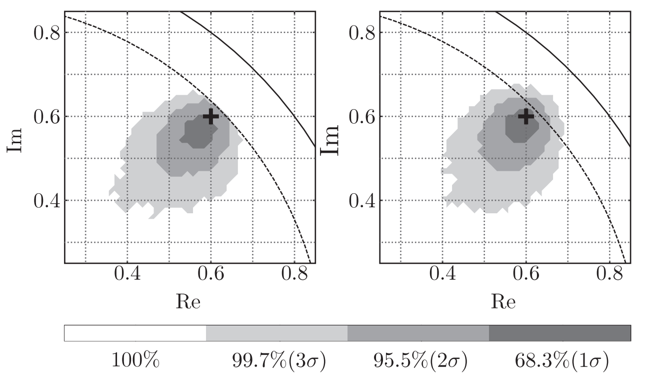

Fig. 1. Poles of constrained estimates (left) and unconstrained estimates (right) generated from 100,000 realizations of data. The shaded regions indicate the concentration of the pole estimates, and '+' marks the location of the system poles. Only the top-half of the complex plane is shown.

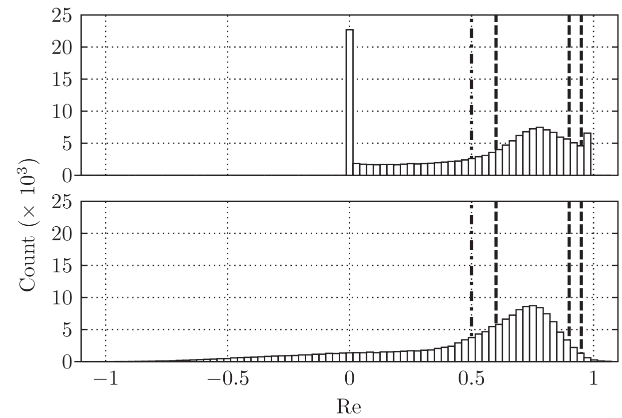

Fig. 2. Histogram of real part of poles of constrained estimates (top) and unconstrained (bottom) generated from 50,000 realizations of data. Dashed lines indicate locations of deterministic system poles, and dash-dot lines indicate location of nondeterministic poles.

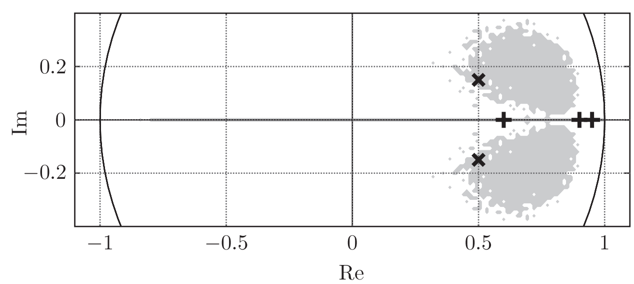

Fig. 3. Poles of unconstrained estimates generated from 45,000 realizations of data. The '+' pair marks the location of the deterministic system poles, and the '×' pair marks the location of the nondeterministic system poles. The shaded region contains 99.7% (3σ) of pole estimates. Regions of lower pole density were not included because they drifted little from the real axis.

Given an unconstrained estimate $\hat{A}^*$ found from (6) and an LMI region described by parameters α and β,

$$\begin{aligned} \text{minimize} \quad & J_X^c(Q,\ P) \\ \text{subject to} \quad & \mathcal{M}(Q,\ P) \geq 0, \\ & P = P^T > 0 \end{aligned} \tag{19}$$

in which

$$J_X^c(Q,\ P) = \left\| Q - A^* P \right\|_F,$$

$$\mathcal{M}(Q,\ P) = \alpha \otimes P + \beta \otimes Q + \beta^T \otimes Q^T.$$

It is again wise to improve numerical conditioning by including the constraint (17). Once Q is found, we then solve for $\hat{A}$ by letting $\hat{A} = QP^{-1}$.

5. Examples

The following examples demonstrate the usefulness of the proposed method. Unconstrained estimates were generated using PI-MOESP as described in Verhaegen and Verdult (2007) with both future and past horizons of 20. The constrained estimates were generated with the same identification method modified to the form of (19). YALMIP (Löfberg, 2004) was used to solve the convex optimization problems with SeDuMi (Sturm, 2001) as the selected solver.

5.1. *Identification with stability constraint*

We first consider a second-order, single-input, single-output system with poles at $0.6 \pm 0.6i$ and a steady-state gain of 1. White noise of variance 1 is added to the input so that deterministic and nondeterministic subsystems have the same poles.

Suppose we desire the poles of the system estimate to lie within the circle of radius 0.87 centered at the origin. The poles of the system are then within the LMI region described by (12) with $\delta_s = 0.13$. Results of both constrained and unconstrained identification methods for 100,000 realizations of datasets with 500 samples each are shown in Fig. 1. The poles of the constrained estimate indeed lie within the described LMI region.

5.2. *Identification with stable, positive, real constraint*

As a second example, consider a third-order, single-input, single-output system with strictly real poles at 0.6, 0.9, and 0.95 and a steady-state gain of 1. A noise signal is added to the output that is generated by white-noise of variance 10 filtered through a system with poles at $0.6 \pm 0.15i$ and a steady-state gain of 1.

Suppose we desire the poles of the system estimate to lie within a region defined as the intersection of the following LMI regions: (i) the circle centered at the origin of radius 0.98, which is the LMI region defined by (12) with $\delta_s = 0.01$; (ii) the plane to the right of the point 0.01 on the real axis, which is the LMI region defined by (13) with $\delta_p = 0.01$; and (iii) the band around the real axis of width 2×10^{-5}, which is the LMI region defined by (14) with $\delta_r = 10^{-5}$. These LMI regions are combined using the identity (11).

The real parts of the poles of both constrained and unconstrained identification methods for 50,000 realizations of datasets with 500 samples each are shown in Fig. 2. Pole locations for the unconstrained estimates are shown in Fig. 3. Pole locations for the constrained estimates are not shown because the imaginary part for all was nearly 0. The poles of the constrained estimates again lie within the given LMI region.

6. Conclusion

We have demonstrated how subspace identification methods may be augmented with convex constraints in the form of linear matrix inequalities to form convex optimization problems that constrain the poles of system estimates to lie within convex regions of the complex plane. Although the method was developed for subspace methods that use an estimate of the extended observability matrix, the modification is generalizable to many other subspace methods. Two simulation examples were presented that demonstrate the utility of constraining poles to lie within such convex regions.

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Daniel N. Miller graduated with a Ph.D. in Mechanical Engineering from the Jacobs School of Engineering at the University of California, San Diego, in 2012. He began his Ph.D. studies in 2009 after several years working in the aerospace industry at General Atomics Aeronautical Systems. He received his B.S. in Mechanical Engineering from the University of California, San Diego, in 2005. He is currently with Vigilent Corporation in El Cerrito, CA developing intelligent energy management systems for data centers. He is a fellow of the Gordon Center for Engineering Leadership and a former Air Force Research Laboratories Space Scholar. His interests are in data-based experimental modeling techniques and robust control systems design.

Raymond A. de Callafon is a Professor with the Department of Mechanical and Aerospace Engineering (MAE) at the University of California, San Diego (UCSD). Raymond de Callafon received his M.Sc. (1992) and his Ph.D. (1998) degrees in Mechanical Engineering from the Delft University of Technology in the Netherlands. Since 1998 he has been with the Dept. of MAE and he is currently directing the System Identification and Control Laboratory (SICL) and is an affiliated faculty of the Center for Magnetic Recording Research (CMRR) directing the CMRR servo laboratory. His research interests include topics in the field of experiment-based approximation modeling, control relevant system identification and recursive/adaptive control.