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MEASUREMENT ERROR ANALYSIS AND STATISTICS

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IN THIS LECTURE:

- **Types of Measurements**
- **Common Terms In Measurements And Error Analysis**
- **Sensor Components in the Measurement Chain**

  transducers, conditioners, converters, amplifiers, filters
- **Error Analysis: Types of errors and Numerical Integration**
- **Error Analysis: Statistical Framework**
- **Propagation of Errors**
- **Linear Regression & Least Squares**
- **Error Analysis in Your Report**

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[4] Steinhaus, H. *Mathematical Snapshots*, 3rd Ed. New York: Dover, 1999.

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TYPES OF MEASUREMENTS

TYPES OF MEASUREMENTS

Direct measurement: comparison of an unknown quantity with a standard assumed to remain constant.

Indirect measurement: characterization of a phenomenon or property in terms of a functional relationship between measured quantities and quantities that are accessible to measurement.

Usually an event cannot be measured directly but causes an electrical or physical signal, that is measured or interpreted. This leads to the concept *of transduction of events and transducers*.

TYPES OF MEASUREMENTS

A transducer converts one form of energy to another form of energy. Usually a transducer is considered to be a device that converts a given form of energy to an electrical signal.

A *fundamental rule* for a transducer: measuring device or method should not alter the event being measured!

$$\textbf{energy drawn from measured system} \leq \frac{1}{100} \textbf{ measuring energy}$$

More on transducers when we discuss sensor components

COMMON TERMS IN MEASUREMENTS AND ERROR ANALYSIS

COMMON TERMS ASSOCIATED WITH MEASUREMENTS AND ERROR:

Accuracy: how closely a measured value approximates the real value.

Precision: how closely measured values cluster around a best estimate of the real value. A precise measurement has *a small standard deviation*, but can be of low accuracy!

Resolution: the level or amplitude of the smallest change in measured value.

In general, instruments change their output in a stepwise fashion, and the relationship between the minimum step size and the full scale output is referred to as *resolution*.

Threshold: the minimum change in input that causes a change in output.

Repeatability: agreement between measurements taken under identical conditions. Given as 1% of Full Scale Output (FSO).

Reproducibility: agreement between different measurements of the same phenomenum.

EXAMPLE: measurement of a temperature

A little history:

Daniel Gabriel Fahrenheit (1686-1736) devoted most of his life to creating precision meteorological instruments. Fahrenheit invented the mercury thermometer in 1714, and later discovered the effect of pressure on the boiling point of liquids. Fahrenheit sought to create a practical temperature scale in which 0 corresponded with the coldest temperature normally encountered in Western Europe and 100 corresponded to the hottest temperature.

In 1742, Anders Celsius (1701-1744) created an inverted centigrade temperature scale in which 100 represented the boiling point of water (373.15 K) and 0 represented the freezing point (273.15 K) at 1 Atm.

Fahrenheit adjusted his temperature scale so that 32 represented the freezing point of water and 212 represented the boiling point of water (373.15 K).

Which temperature scale makes more sense?

EXAMPLE: measurement of temperature of freezing water

Let the actual temperature be 32°F, consider the following measurements:

32.11	36.21	32.5	34.51	32.11
31.92	35.93	31.5	30.05	31.92
32.01	36.03	32.0	33.05	32.01
32.03	35.99	32.5	31.12	42.03
31.98	36.02	31.5	30.87	31.98
31.89	35.98	32.0	33.65	31.89
accurate and precise with relative high resolution	precise with relative high resolution but not accurate	accurate but less precise due to low resolution of 0.5°F	less precise but most likely accurate with relative high resolution	?

EXAMPLE: measurement of a temperature (SOLUTION)

Let the actual temperature be 32°F, consider the following measurements:

32.11	36.21	32.5	34.51	32.11
31.92	35.93	31.5	30.05	31.92
32.01	36.03	32.0	33.05	32.01
32.03	35.99	32.5	31.12	42.03
31.98	36.02	31.5	30.87	31.98
31.89	35.98	32.0	33.65	31.89
accurate and precise with relative high resolution	precise with relative high resolution but not accurate	accurate but less precise due to low resolution of 0.5°F	less precise but most likely accurate with relative high resolution	accurate, not precise (due to outlier at 42.03) with high resolution

Error characterization is a combination of

- mean value (accuracy) and variation (precision, resolution)

COMMON TERMS ASSOCIATED WITH MEASUREMENTS AND ERROR:

Hysteresis: difference in output between measurements of increasing and decreasing input.

Hysteresis will yield different measurements in case the input increases or decreases and this error is bad for measurements!

Linearity: deviation of a response curve (or calibration curve) from a straight line, given in ± % FSO.

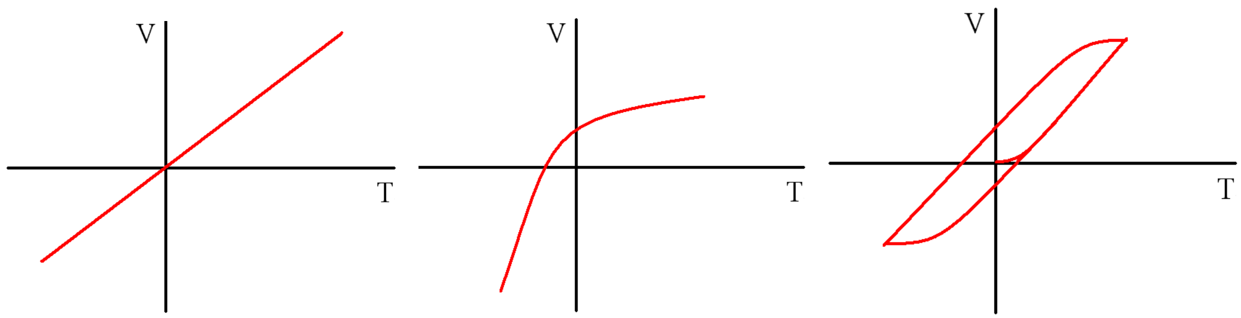
Linearity makes it easy to find a relation between measured quantity and resulting output signal: a linear relation.

Conformance: degree of correspondence *between nonlinear input/output relationship* and theoretical curve.

Offset: output for 0 input level.

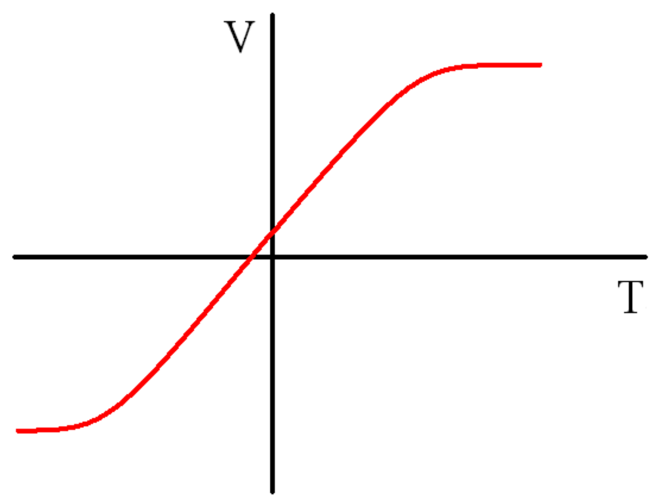
EXAMPLE: measuring temperature with a thermocouple

Thermocouple generates voltage depending on temperature, then the following characteristics indicate:



linearity without offset | non-linearity with offset | hysteresis

Most (good) sensors have the following static characteristic curve:



I.E. *Linear over a certain range with a small offset.*

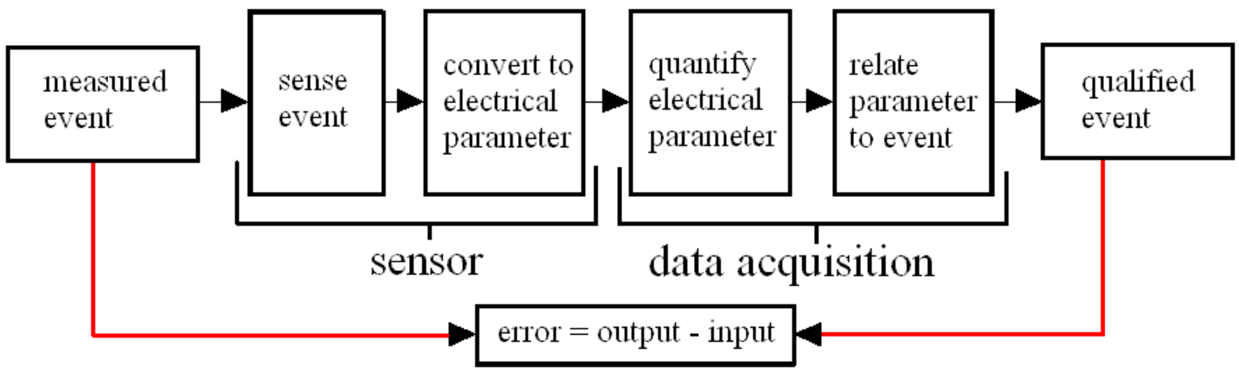
SENSOR COMPONENTS IN THE MEASUREMENT CHAIN

SENSOR COMPONENTS INTHE MEASUREMENT CHAIN

Many quantities can be measured *directly*, most events however are measured *indirectly* (indirect measurement)

Examples are: temperature, pressure, force, acceleration, speed, flow rate, etc.

The general process of an *indirect measurement* is shown schematically in the following figure:



most important element: *the sensor* (transduces and conditions the `event to measure', so it can be processed by *the data acquisition system*)

SENSOR COMPONENTS

Measuring a signal often involves many **components**:

- **transducer** (sometimes also called, `the sensor')

  converts one form of energy to another form of energy (electricity)

- **power source**

  needed to supply (electric) energy

- **amplifiers**

  to amplify the signals being measured before using them

- **filters**

  to get rid of unwanted (high or low frequent) disturbances

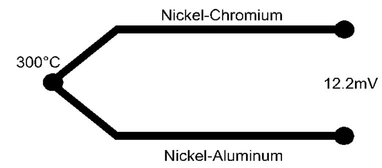
- **possible multiplexers and converters**

  in case of digital data acquisition systems

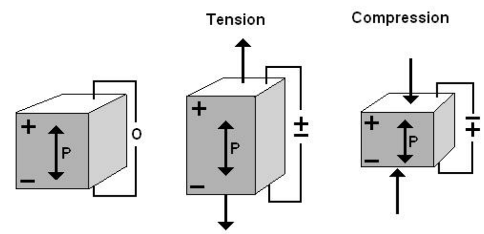
signal conditioning (power source, amplifiers, filters)

Following is summary of abovementioned components to review the measurement chain.

SENSOR COMPONENTS

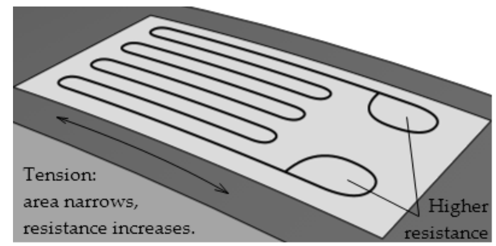


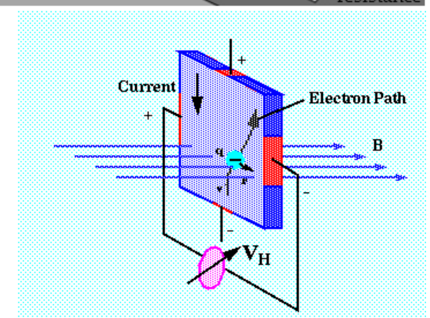
Transducers/sensors



Active (generate their own energy). Examples:

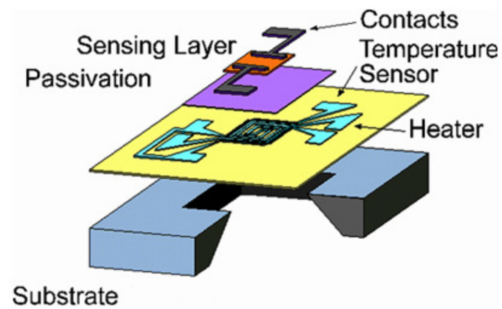
- thermocouples
- piezoelectric sensors





Passive (require additional energy source). Examples:

- strain gage
- Hall effect sensor
- platinum resistance temp. gage



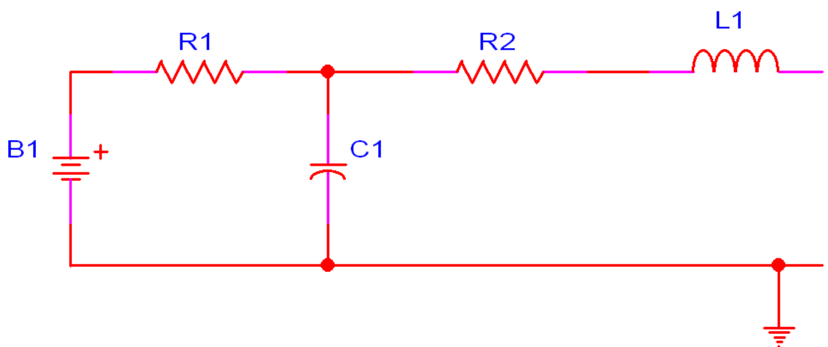
SENSOR COMPONENTS

Power source

Provides excitation for passive gages, suppresses sensor zero.

Characteristics:

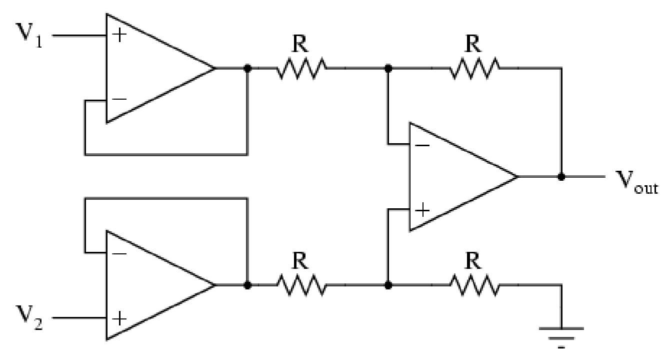
(1) operating mode (const. Voltage or const. Current)
(2) line and load regulation spec.
(3) ripple on source (Voltage or current ripple)
(4) zero suppression range
(5) loading or stability



SENSOR COMPONENTS

Amplifier

- typically used to maximize measurement resolution by scaling sensor voltage range to full analog-to-digital converter (ADC) range
- compensates for differences between sensor's zero reference potential and data acquisition system's zero potential -- generally a differential amplifier acts on difference between input lead.



Characteristics:

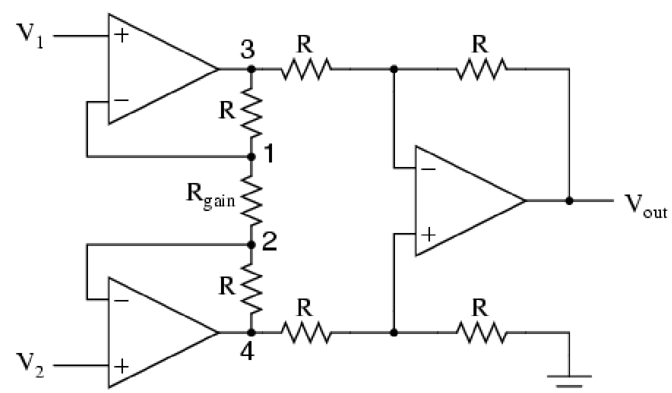
(1) Input electrical characteristics
(*impedance, capacitance)*

(2) Design type
(*single ended, differential, instrumental)*

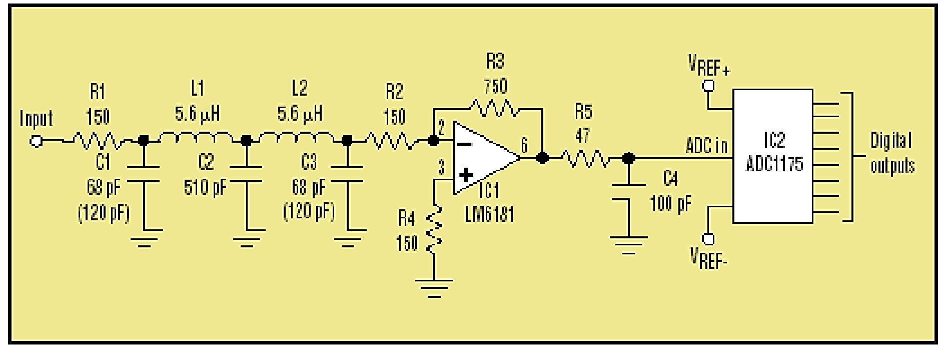
(3) Dynamic parameters
(*bandwidth, slew rate, settling time, overshoot)*

(4) Max. input voltage and gain

$$A_V = (1 + \frac{2R}{R_{gain}})$$

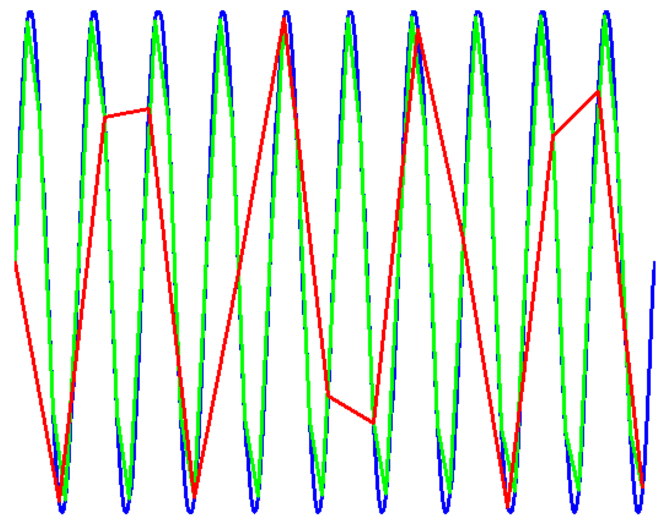


SENSOR COMPONENTS



Anti-Aliasing Filter

- Limits the frequency content of a **sampled signal** typically by a low pass filtering. Example: eliminate (alternate) high frequency noise by dedicated low pass filter.
- The term "anti-aliasing" is used because failure to select the correct sampling rate creates **aliasing** effects.
  aliasing = slow sampling of a high frequency signal makes it appear as a low frequency signal.



Sine signal (blue) sampled at a high frequency (green) and a low frequency (red)

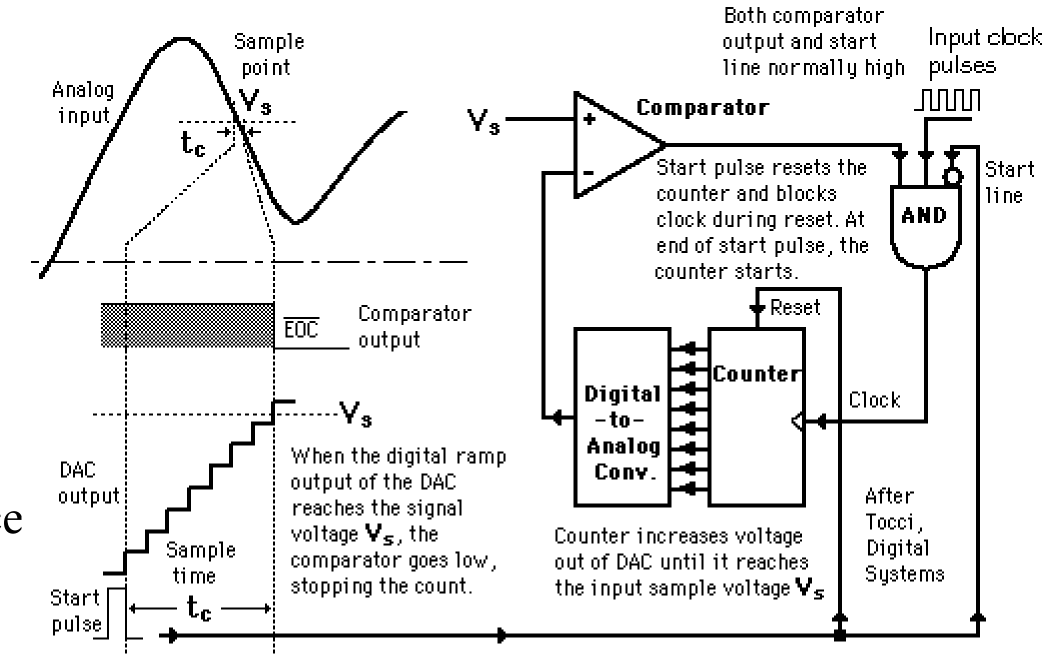
Characteristics:

(1) filter type (*Bessel, Butterworth, Chebyshev)*

(2) pass band or stop band parameters

(3) dynamic parameters (*settling time, overshoot)*

SENSOR COMPONENTS



Analog-to-Digital Converter (ADC)

converts an analog signal (voltage) to digital information (numbers) by a direct or indirect comparison of the signal to a known reference

Characteristics:

(1) Conversion techniques:

Digital Ramp (see figure)

, Successive Approximation

or *Flash ADC*

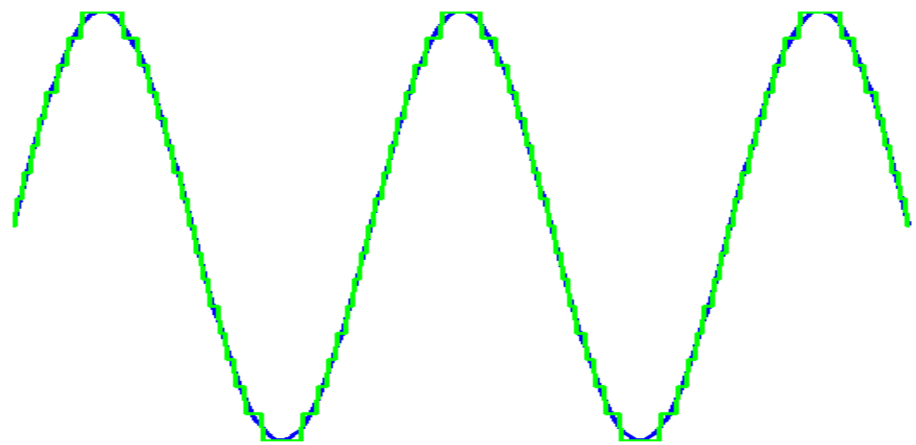
(2) Conversion rate

points/second

(3) Full scale conversion voltage

(4) Resolution

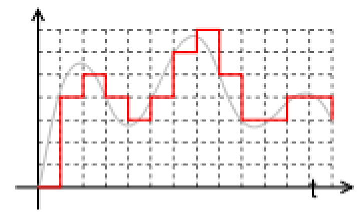
number of bits for full scale, ex: 8, 10, 12 bits will yield 2^8, 2^{10}, 2^{12} parts of full scale



Sine signal (blue) sampled at a high frequency with a 4 bit resolution (green)

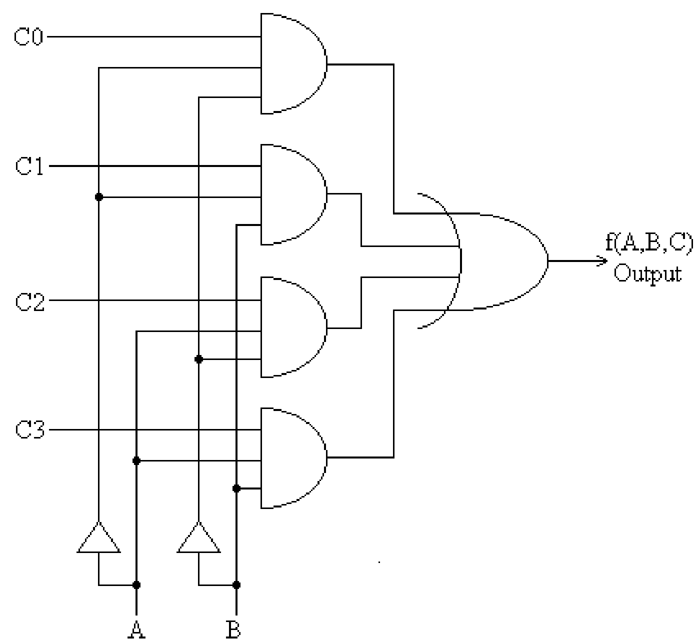
SENSOR COMPONENTS

Zero Order Hold Circuitry



Holds signal constant during the time the ADC takes to digitize an analog signal and/or to convert a digital signal back into a continuous signal.

Multiplexer Circuitry



Switch to connect N different sensor input channels to one data measuring equipment (ADC).

Characteristics:

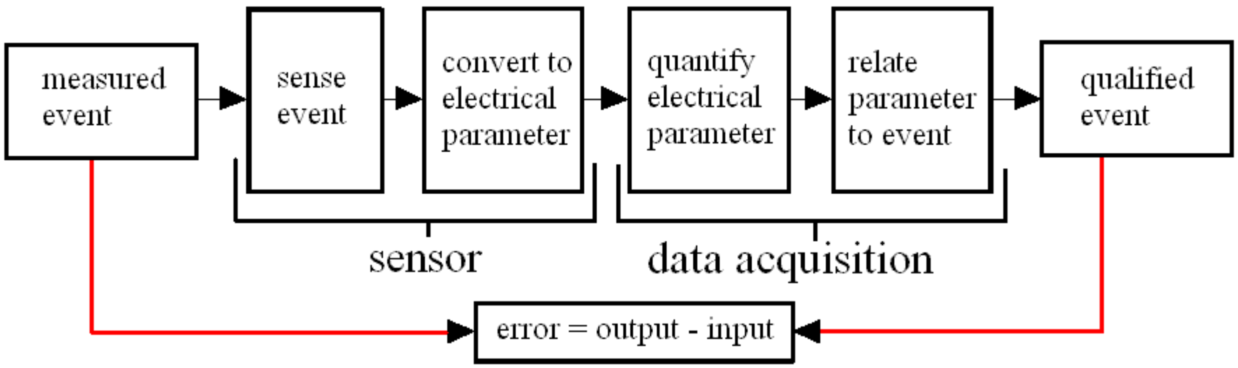
(1) switching rate

(2) cross-talk.

ERROR ANALYSIS: TYPES OF ERRORS AND NUMERICAL INTEGRATION

ERROR ANALYSIS

Error is defined as *the difference between event to measure and quantified event.* This error can characterized as the absolute difference.



Usually, relative measures are being used:

$$\text{Error in \% of reading} \equiv \frac{\text{Error}}{\text{System Output}} \times 100$$

$$\text{Error in \% of Full Scale (FS)} \equiv \frac{\text{Error}}{\text{FS Output}} \times 100$$

ERROR ANALYSIS

<ins>(FOUR) TYPES OF ERRORS</ins>:

(1) **Intrinsic Errors**: errors inherent with the measurement chain

- (a) sensor accuracy (or “inaccuracy”)
- (b) linearity and conformance
- (c) hysteresis
- (d) offset
- (e) noise
- (f) repeatability
- (g) resolution
- (h) threshold

All these errors are due to the *different components in the measurement chain*.

ERROR ANALYSIS

(FOUR) TYPES OF ERRORS:

(2) **Application Related Errors**: errors associated with the use of sensors.

(a) Spatial errors

(proper location of sensor(s) or use of multiple sensors)

(b) Interaction errors

(size of sensor if too big: affects measurement)

(c) Probe errors

(orientation of sensor)

(d) Temperature effects

(changes resistance)

ERROR ANALYSIS

(FOUR) TYPES OF ERRORS:

(3) **Interface Errors**: errors due to interfacing measurement chain components

(a) Cabling (resistance, impedance, capacitance, etc.)

(b) Loading (electrical or mechanical load)

(c) Common mode voltage

(d) Static cross-talk (multiplexed systems)

(4) **Sampling and Approximation Errors**: errors due discretization of analog signals

(a) Finite time sampling

(b) Sampling distribution error (improper sampling rate)

(c) Stabilization error (due to sensor response time)

(d) Approximation error (due to finite # points, **integration**)

ERRORS ANALYSIS: NUMERICAL INTEGRATION (Riemann sums)

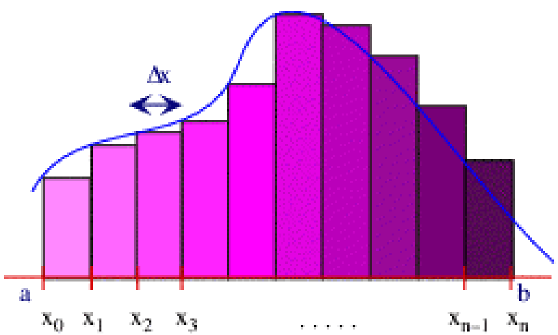
Consider a measurement that is found via an integration:

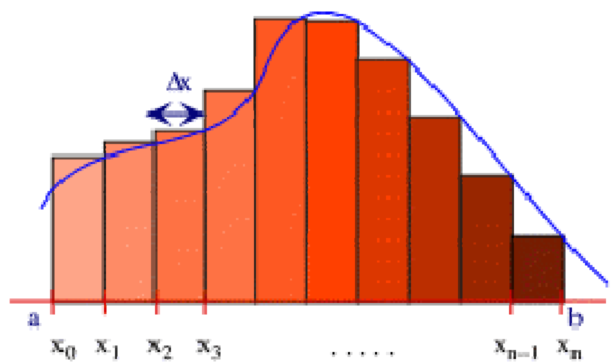
$$\int_a^b f(x)\,dx$$

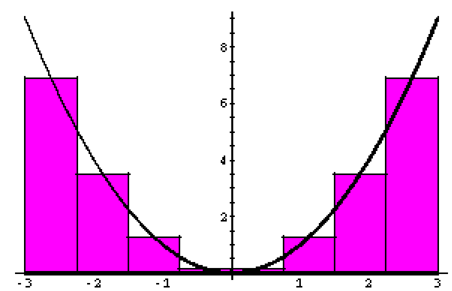
where x is measured at n discrete points a_i , i=1,2,…,n on the interval [a,b], then integration can be done via:

(1) Riemann sums using left, right or middle points

$$L_n = \sum_{i=1}^{n} f(a_{i-1})\,\Delta x \qquad R_n = \sum_{i=1}^{n} f(a_i)\,\Delta x \qquad M_n = \sum_{i=1}^{n} f\left(\frac{a_{i-1}+a_i}{2}\right)\Delta x$$







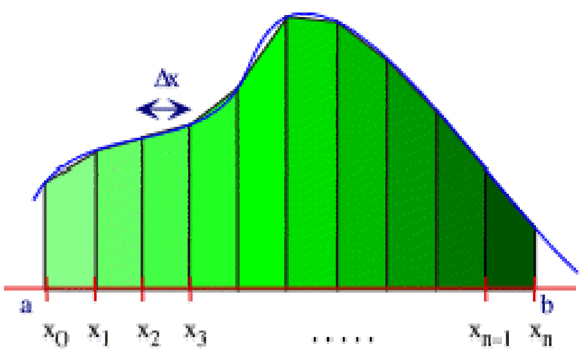
ERRORS ANALYSIS: NUMERICAL INTEGRATION (trapezoidal and Simpson)

Consider a measurement that is found via an integration:

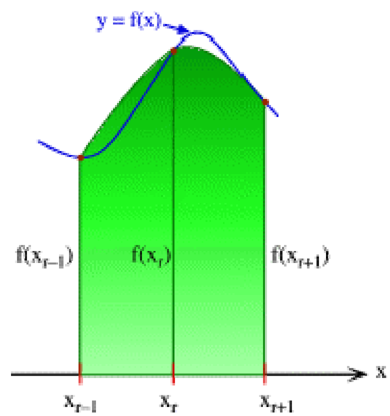
$$\int_a^b f(x)\, dx$$

where x is measured at n discrete points a_i , i=1,2,…,n on the interval [a,b], then integration can also be done via:

(2) Combination : the Trapezoidal rule (line or 1st order approximation) or Simpson's rule (parabolic or 2nd order approximation)



$$T_n = \frac{L_n + R_n}{2}$$



$$S_{2n} = \frac{T_n + 2M_n}{3}$$

Pictures courtesy of *Stefan Warner and Steven R. Costenoble, Dept. of Mathematics, Hofstar University* and *Lawrence S. Hush, Dept. of Mathematics, University of Tennessee, Knoxville*.

ERRORS ANALYSIS: NUMERICAL INTEGRATION (Results)

Each integration method introduces errors that can be bounded:

Trapezoidal Rule Error Bound: Consider **n** points over **[a,b]** and suppose that the second derivative of **f** is continuous on **[a, b]** with $|f^{(2)}(x)| \leq M$ for all **x** in **[a, b]**. Then

$$|e_T| \leq \frac{M\,(b-a)^3}{12\,n^2}$$

Midpoint Rule Error Bound: Consider **n** points over **[a,b]** and suppose that the second derivative of **f** is continuous on **[a, b]** with $|f^{(2)}(x)| \leq M$ for all **x** in **[a, b]**. Then

$$|e_M| \leq \frac{M\,(b-a)^3}{24\,n^2}$$

Simpson's Rule Error Bound: Consider **n** points over **[a,b]** and suppose that the fourth derivative of **f** is continuous on **[a, b]** with $|f^{(4)}(x)| \leq M$ for all **x** in **[a, b]**. Then

$$|e_S| \leq \frac{M\,(b-a)^5}{180\,n^4}$$

ERROR ANALYSIS: STATISTICAL FRAMEWORK

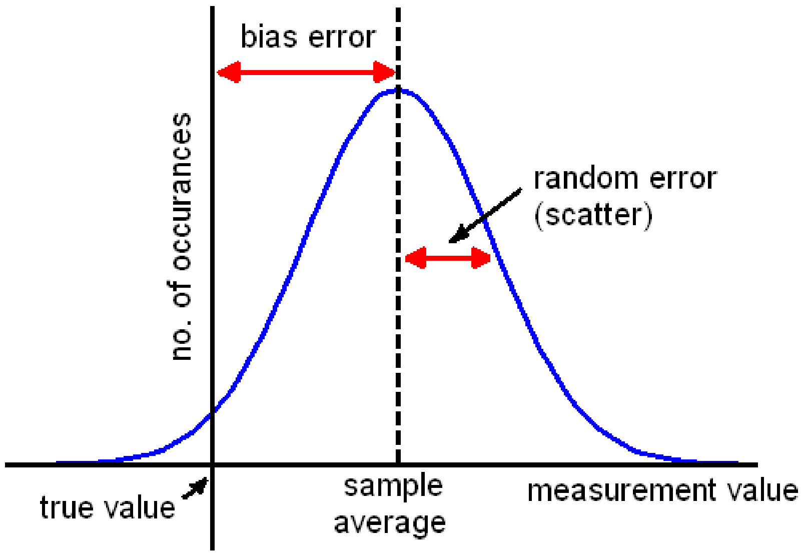
ERROR ANALYSIS (Statistical Framework)

There are two essential components of measurement error:

(a) **Bias error** (related to the mean value or accuracy)

(b) **Random error** (related to the variation in data due to precision and resolution)

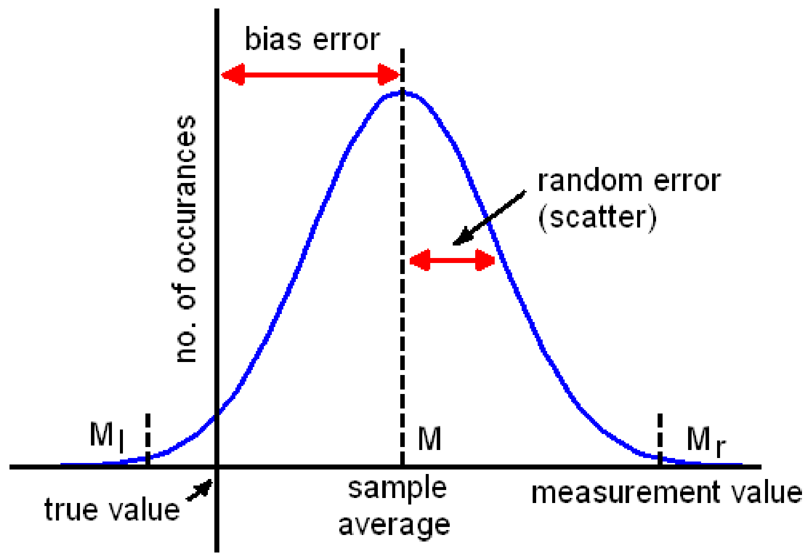
For the case of a *constant value of the event to measure*, these two main components are illustrated in the figure of the **probability density function**:



probability density function

ERROR ANALYSIS (Statistical Framework)

The *fixed* difference between the true value and the average of all repeated measurements, $\overline{M}$, is the **bias error**, whereas the deviation of individual measurements from the average $\overline{M}$ are the **random errors**. Bounds M_l and M_r indicate **confidence interval** for $\overline{M}$



The "bell shaped" **probability density function** above shows that measurements near the average are more probable than far from it.

It gives insight in the `spread' of the random effect of the measurements.

ERROR ANALYSIS (Statistical Framework)

The **statistical framework** allows the uncertainty associated with any measurement to be described in a **probability density function** or a **confidence interval** that are functions of both ***bias*** **and** ***random*** **errors**.

Example

To estimate the bias error, we apply a constant input and take n measurements. Then the bias error can then be *estimated* via

$$B_i = \bar{e} = \frac{\Sigma e_i}{n}$$

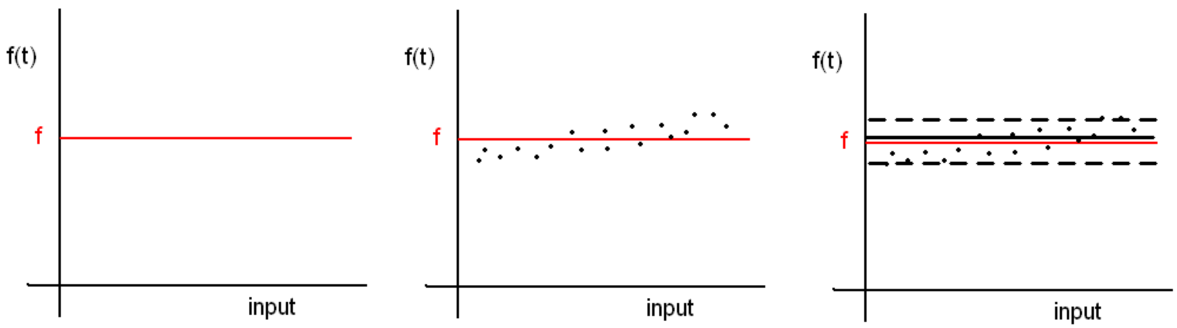
which is nothing else than simply taking the **average of the samples/measurements**.

Since this estimate is obtained from n samples:

- a probable band for the true average error can be established using the t-statistic (t-distribution function)
- With information on probability density function we can formulate confidence intervals for our measurement.

SOME STATISTICS

All measurements are in error, and the purpose of error analysis is to quantify these errors. For random errors we can use statistics.



The idea is to characterize the bias (*mean*) and the spread (*variance*) of the measurement to characterize *the quality of the measurement.*

If we use n observations, the error at the ith observation is

$$e_i = f(iT) - f_1$$

where f_1 is the *known* input or *know* quantity to measure

SOME STATISTICS - terminology

Population = all elements of measured quantity – finite (N) or infinite (∞).

Sample = n measurements of the population. Sample is representative of population if:

(a) sample can be characterized

(b) relationship between sample parameters and population parameters is *known*.

Central tendency (of population)

(a) median

(b) mean, μ

Dispersion parameters (of population)

(a) deviation: $d_i = e_i - \mu$, μ = population mean

(b) variance: $\sigma^2 = \frac{\Sigma d_i^2}{N}$

(c) standard deviation: σ

SOME STATISTICS – mean/average and median

For the whole population of measurements we have the population parameters based on a total of N measurements. For a sample of n measurements (since the whole population may not be available) we have sample parameters. In general, one wants to obtain the population statistics from the sample statistics.

A summary of those is as follows:

Central tendency	Population	Sample	
Mean/average:	$\mu \equiv \frac{\Sigma e_i}{N}$	$\bar{e} \equiv \frac{\Sigma e_i}{n}$ $\Rightarrow \Sigma(e_i - \bar{e}) = 0$	Also called **Sample Mean**
Median:	middle pt., i.e. same # of pts. above and below median		

SOME STATISTICS – deviation, variance and standard deviation

Dispersion tendency	Population	Sample	
Deviation:	$d_i = e_i - \mu$	$d_i = e_i - \bar{e}$	
However,	$\Sigma d_i = 0$	$\Sigma d_i = 0$	
so we can define			
Variance:	$\sigma^2 \equiv \frac{\Sigma d_i^2}{N} = \frac{\Sigma (e_i - \mu)^2}{N}$	$\hat{\sigma}^2 \equiv \frac{\Sigma (e_i - \bar{e})}{n}$	Also called **Sample Variance**
Std. Deviation:	$\sigma \equiv \sqrt{\frac{\Sigma (e_i - \mu)^2}{N}}$	$\hat{\sigma} \equiv \sqrt{\frac{\Sigma (e_i - \bar{e})^2}{n}}$	

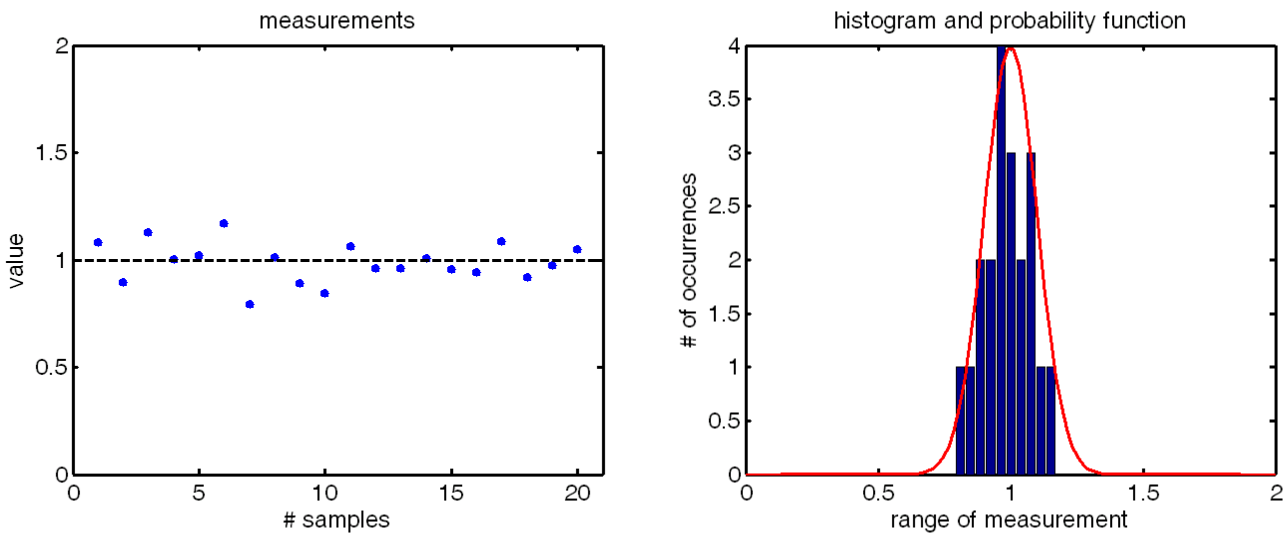
To determine the sample standard deviation $\hat{\sigma}$ *during* a measurement, without having to store all observations, we can use the **recursive formulae**:

$$\hat{\sigma} = \left(\frac{\Sigma e_i^2}{w} - \bar{e}^2 \right)^{1/2} = \left(\frac{\Sigma e_i^2 - \Sigma e_i}{n} \right)^{1/2} .$$

SOME STATISTICS – estimate of probability density function

- The statistical properties of a population are fully characterized by its **probability density function** (that may be a function of μ and σ as defined before)
- If the frequency (no. of occurrences) of the errors is plotted as a function of error amplitude, *an estimate of* ***the probability density function*** is obtained!

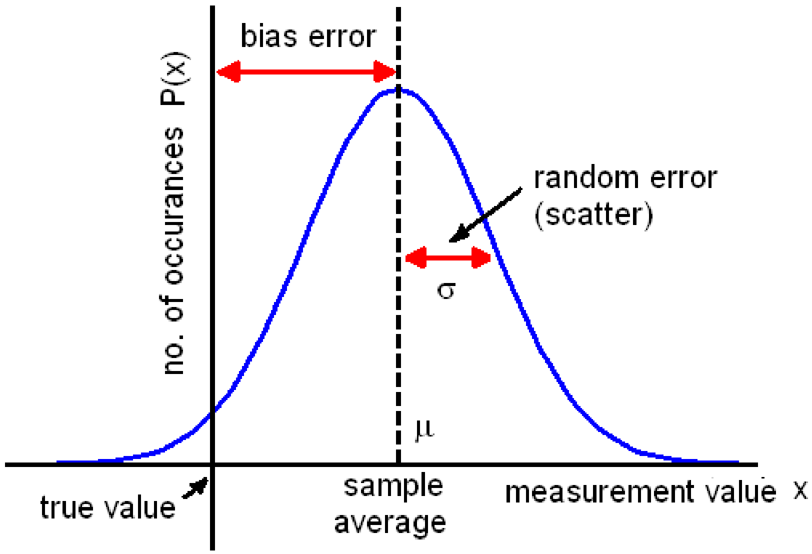
Example: measurement with a mean value $\mu=1$ and a standard deviation $\sigma=0.1$:



Histogram (blue) gives estimate of the **probability density function (red)** and shows mean and spread of measurement errors.

SOME STATISTICS – normal or Gaussian distribution

Many populations exhibit a **Normal (or Gaussian) distribution** with a **Gaussian Probability Density Function**. For such a distribution, the probability distribution has a `bell shaped' character.



$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$P(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz.$$

<u>Characteristics of Gaussian or Normal distribution</u>:

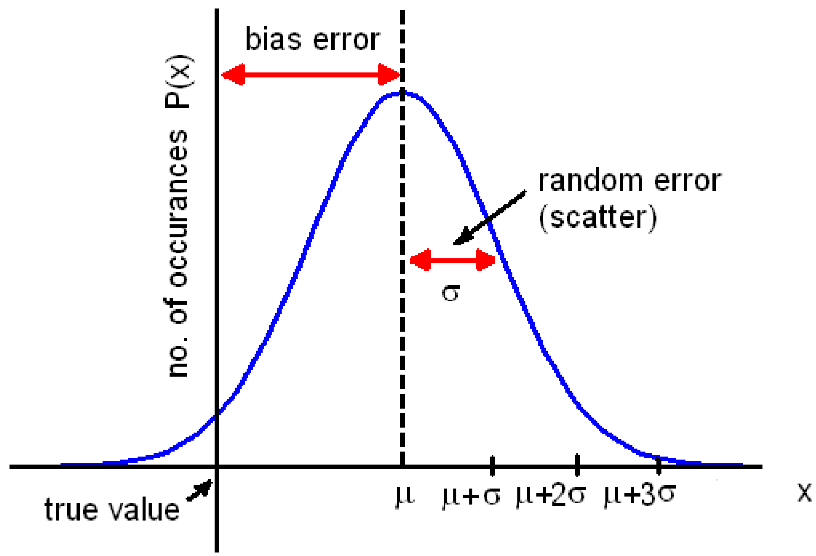
(1) Fully characterized μ and σ

(2) Symmetric around the mean μ

(3) area under curve = 1 (as for ALL probability density functions!)

(4) probability density approaches 0 for larger values and σ *determines "how fast"*

SOME STATISTICS – normal or Gaussian distribution



The probability that a measurement occurs between $\pm\sigma$, $\pm 2\sigma$, $\pm 3\sigma$ is the following:

$$P(-\sigma \le e \le \sigma) = 0.683$$

$$P(-2\sigma \le e \le 2\sigma) = 0.954$$

$$P(-3\sigma \le e \le 3\sigma) = 0.997$$

Thus, for example, there is a 99.7% probability that the error for a *normally* distributed population (or sample) is within 3 standard deviations from the mean. Thus, **the standard deviation is a measure of *dispersion* used with the normal distribution**.

SOME STATISTICS – normalized distribution

Normalized z-variant:

In order to calculate the probability that the *population* error lies between e_1 and e_2 we use **Standardized Normal/Gaussian Distribution** characterized by $\mu = 0$ and $\sigma = 1$.

Any measurement/variable e with a Normal/Gaussian Distribution with μ and σ can be given a **Standardized Normal/Gaussian Distribution** characterized by $\mu = 0$ and $\sigma = 1$:

define $z_i = \dfrac{e_i - \mu}{\sigma}$ as a `new' random variable and called the **Normalized z-variant**

this gives $\mu_z = 0, \;\; \sigma_z^2 = 1$, so *normalized* and then the **normalized** Gaussian probability density function $f(z) = \frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz$ (tabulated on page **59**) can be used for calculations!

SOME STATISTICS – example of confidence intervals via normalized distribution

Example

20 observations are made for an instrument calibration. Assume

$$\bar{e} = \mu = 0.0117$$

$$\hat{\sigma}^2 = \sigma^2 = 0.0119$$

$$\Rightarrow \quad \sigma = 0.1091$$

Determine the 95% CONFIDENCE INTERVAL, This means, compute $e_{min} = e_1$ and $e_{max} = e_2$ such that 95% of the observations will lie in this interval.

For 95% probability see Standard Normalized Distribution table on page **59**. Table lists only probability for P(0<z<a), but with symmetry of P(z) we now 95/2 = 47.5% = 0.475. From tale we observe *a*=1.96, so for normalized z-variant we have

$$\Rightarrow z = \pm 1.96$$

$$z = \frac{e_i - \mu}{\sigma} \Rightarrow e_i \sigma z_i + \mu \Rightarrow \qquad e_1 = -0.202$$

and $e_2 = 0.225$

SOME STATISTICS – properties of mean estimation

Take a set of *n* samples from population, with sample averages $\bar{e}_j$. Now, take the average of *all* sample means. This average is called the ***expected value of the sample mean***, and is equal to μ:

$$\bar{\bar{e}} = E(\bar{e}) = \mu .$$

Now, the standard deviation of the distribution of sample means is the ***standard deviation of the sample mean***:

$$\sigma_{\bar{e}} = \frac{\sigma}{\sqrt{n}}\left(\frac{N-n}{N-1}\right)^{1/2} \text{ "finite population correction"}$$

If $n \leq \frac{N}{10}$ or $N \to \infty$ then $\sigma_{\bar{e}} \approx \frac{\sigma}{\sqrt{n}} \Rightarrow \sigma = \sqrt{n}\hat{\sigma}$

Interesting Fact: for $n \geq 30$ $\bar{e}$ is *normally distributed* even if the population is *not* (**central limit theorem**). Since $\bar{e}$ is *normally distributed*, we can write (normalization):

$$z = \frac{\bar{e} - \mu}{\sigma_{\bar{e}}} = \frac{\bar{e} - \mu}{\sigma / \sqrt{n}} .$$

SOME STATISTICS – example of mean estimation

<u>Example</u>

Compute 95% CONFIDENCE INTERVAL for μ if you know that the sample mean $\bar{e} = 0.0117$, on the basis of n=20 samples from a population with $\sigma = 0.1$.

> 95% $\Rightarrow$ 95/2=47.5% = 0.475 for P(0<z<a)
>
> with z-table on page **59**, P(0<z<a) = 0.475 for a=1.96
>
> $\Rightarrow z = \pm 1.96$
>
> **Work out the problem yourself!**

NOTE: The above example assumes knowledge of the population standard deviation σ. Usually, this is not available, but we are estimating this via the **sample variance**…

SOME STATISTICS – properties of variance estimation

It can be shown that if all possible samples of size *n* are taken from a population and each of these is used to compute the sample variance $\hat{\sigma}^2$, then the ***expected value of the sample variance is***:

$$E\left(\hat{\sigma}^2\right) = \frac{n-1}{n}\sigma^2 = \sigma^2 - \frac{\sigma^2}{n}$$

thus the ***expected value of sample variance*** is not σ^2, but $\frac{\sigma^2}{n}$ less than the population variance! For **small n there is an error**, which is adjusted by ***adjusted sample variance***:

$$s^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{\Sigma\left(e_i - \bar{e}\right)^2}{n-1}$$

So, instead of $\hat{\sigma}^2 \equiv \frac{\Sigma\left(e_i - \bar{e}\right)}{n}$ (as we saw before) we use the estimate:

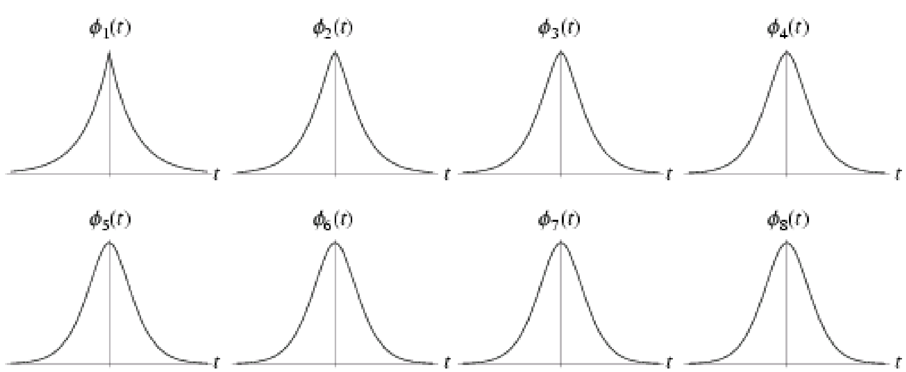
$$s^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{\Sigma\left(e_i - \bar{e}\right)^2}{n-1}$$

SOME STATISTICS – t-distribution

Sample average $\overline{e}$ with ***adjusted sample standard deviation*** s can be related to the population mean μ via a so-called "t-distribution": $t = \dfrac{\bar{e}_i - \mu}{\frac{s}{\sqrt{n}}}$, $s = \left[\dfrac{\Sigma(e_i - \bar{e})^2}{n-1}\right]^{1/2}$

NOTES:

- This is again a *normalization* of the statistical variable e (your measurement)
- To get the normalized t-distribution, one has to subtract the mean value μ and divide by the adjusted sample standard deviation s and multiply by square root of n.
- The degrees of freedom (d.f.) defined by d.f. = $n-1$, where n = number of samples, determines the shape of the t-probability density function:



shape of t-probability density function $\Phi_{df}(t)$, where df = degrees of freedom

SOME STATISTICS – t-distribution

General trend for t-distribution:

The larger the d.f. = *n-1* (degrees of freedom), the closer the t-distribution resembles a (normalized) Normal distribution. Hence for large n ($n>30$) you can simply work with Normal distribution!

Table

For smaller values (typically d.f.<30), the probability density function of the t distribution differs from the normal distribution, but t-distribution is a standard distribution that is listed in tables. See (inverse) t-distribution table on page **60**.

SOME STATISTICS – t-distribution

Example:

Compute 95% CONFIDENCE INTERVAL for μ if you know that: the sample mean $\bar{e} = 0.012$ and adjusted sample variance s = 0.11 on the basis of n=20 samples from a population.

With this information we have d.f. = 19

95% $\Rightarrow$ one tail of Inverse Standard t-distribution is 2.5% $\Rightarrow$ need $t_{.025}$ $\Rightarrow$ from the t-table on page **60** we see: $t_i = \pm 2.093$. Note: this is *different* from the $z_i = \pm 1.96$ we found when using a Standard Normal Distribution!

$$\Rightarrow \quad -2.093 < \frac{\bar{e} - \mu}{s / \sqrt{n}} < 2.093$$

$$\therefore -0.032 < \mu < 0.056 .$$

SOME STATISTICS – properties of variance estimation

Similarly, the population variance or standard deviation σ can be related to the variance estimate (sample variance) by a so-called chi-square (x^2) distribution. This can be done can for *normally distributed* samples:

$$x^2 = \frac{\Sigma(e_i - \bar{e})^2}{\sigma^2} \text{ used with d.f.} = n-1.$$

With given d.f., from the x^2 distribution we have

$$a > x^2 > b \Rightarrow a > \frac{\Sigma(e_i - \bar{e})^2}{\sigma^2} > b$$

$$\therefore \frac{\Sigma(e_i - \bar{e})}{a} < \sigma^2 < \frac{\Sigma(e_i - \bar{e})^2}{b}$$

SOME STATISTICS – summary

Sample average: $\bar{e} = \frac{\Sigma e_i}{n}$

Sample variance: $\hat{\sigma}^2 = \frac{\Sigma\left(e_i - \bar{e}\right)^2}{n}$

Adjusted sample variance: $s^2 = \frac{\Sigma\left(e_i - \bar{e}\right)^2}{n-1}$

Degrees of freedom: $df = n - 1$

Estimate for μ: $\bar{e} - t_{\alpha,df}\left(\frac{s}{\sqrt{n}}\right) < \mu < \bar{e} + t_{\alpha,df}\left(\frac{s}{\sqrt{n}}\right)$

Estimate for σ^2 $\frac{\Sigma\left(e_i - \bar{e}\right)^2}{a} < \sigma^2 < \frac{\Sigma\left(e_i - \bar{e}\right)^2}{b}$

where α = probability

$d.f.$ = degree of freedom

$t_{\alpha,df}$ = t-distribution for α, df

a, b from x^2 distribution

PROPAGATION OF RANDOM ERRORS

Q: What is the *total* accuracy δF of a measured system if the measurement depends on n parameters being measured, i.e.:

$$F = f(m_1, m_2, \ldots, m_n)$$

A: Form a Taylor-series expansion we have

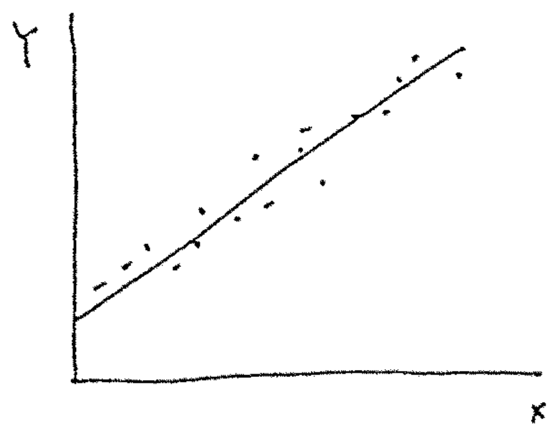
$$\partial F = \frac{\partial f}{\partial m_1}\partial m_1 + \frac{\partial f}{\partial m_2}\partial m_2 + \ldots + \frac{\partial f}{\partial m_n}\partial m_n$$

Since the ∂m_i 's are random variables, we can use:

$$\delta F = \pm\left[\left(\frac{\partial f}{\partial m_1}\right)^2 \delta m_1^2 + \left(\frac{\partial f}{\partial m_2}\right)^2 \delta m_2^2 + \ldots + \left(\frac{\partial f}{\partial m_n}\right)^2 \delta m_n^2\right]^{1/2}$$

thus, one can add errors add in an RMS (root mean square) sense!

LINEAR REGRESSION



Assumptions:
(1) more measurements than needed for curve
(2) only random errors
(3) independent measurements

$Y = aX + b$ = regression equation
a, b = regression coefficients.

The idea is to capture the measurements in a linear regression model $Y = aX + b$, where a, b are the regression coefficients to be determined.

Methods for getting $Y = aX + b$:

(1) graphic method ("eyeball"): works OK, but subjective.

(2) least squares

LINEAR REGRESSION – least squares

Assume errors occurred only in the Y measurement, then measurements should satisfy:

$$Y_1 = a\,X_1 + b + E_1$$

$$Y_2 = a\,X_2 + b + E_2$$

$$\vdots$$

$$Y_n = a\,X_n + b + E_n$$

Short hand notation in matrix form:

$$Y = X\,[a\ b]^T + E$$

With

$$Y = [\ Y_1\ ;\ Y_2\ ;\ ..\ ;\ Y_n\ ] := [\ Y_1\ Y_2\ ..\ Y_n\ ]^T$$

$$X = [X_1\ 1\ ;\ X_2\ 1\ ;\ ..\ ;\ X_n\ 1]$$

$$E = [\ E_1\ ;\ E_2\ ;\ ..\ ;\ E_n\ ]$$

LINEAR REGRESSION – least squares

Least Squares Estimation:

Find the parameter [a b] such that $||E||_2$ is minimized, where

$$||E||_2 = tr\{E^T E\}$$

Solution to Least Squares:

1. Convex optimization (minimum of a quadratic function)
2. Orthogonal projection (to find minimum error norm)

Both solutions are the same…

LINEAR REGRESSION – least squares

Consider

$$Y = X\,[a\ b]^T + E$$

To find the parameter [a b] **via optimization** of $||E||_2 = tr\{E^T E\}$:

$$tr\{E^T E\} = tr\{[Y^T - X^T\,[a\ b]][Y - X\,[a\ b]^T]\}$$

LINEAR REGRESSION – least squares

Setting $d\|E\|_2/d[a\ b] = 0$ will give the optimal solution:

$$d\|E\|_2/d[a\ b] = [Y^T - [a\ b]\ X^T]X = Y^T X - [a\ b]\ X^T X = 0$$

Solving for [a b] yields:

$$\mathbf{[a\ b] = Y^T X\ [X^T X]^{-1}}$$

To find the parameter [a b] **via orthogonal projection**, smallest distance E between $Y = X\ [a\ b]^T$ and $Y = X\ [a\ b]^T + E$ is found by orthogonal projection of Y onto $X\ [a\ b]^T$. This projection makes E orthogonal to X so choose error E such that $X^T E = 0$:

$$X^T Y = X^T X\ [a\ b]^T + X^T E = X^T X\ [a\ b]^T$$

making

$$\mathbf{[a\ b] = Y^T X\ [X^T X]^{-1}}$$

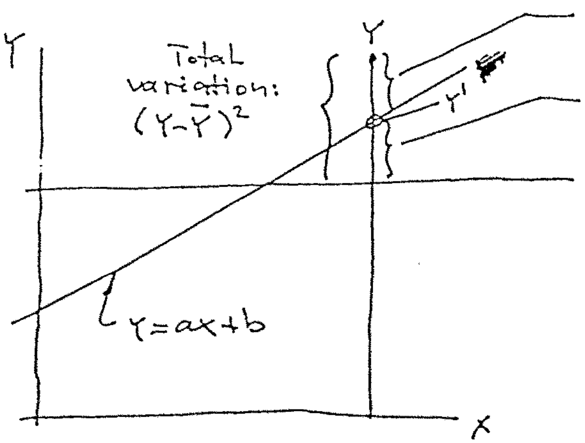
LINEAR REGRESSION – least squares

Standard error of the estimate: $S_{YX} = \sqrt{\dfrac{\Sigma y^2 - b\Sigma xy}{n-2}}$

Correlation coefficient: $r = \sqrt{\dfrac{\Sigma xy}{\Sigma x^2 \Sigma y^2}}$

The correlation coefficient for a population infers correlation between *X* and *Y*, but not for samples, except to within some uncertainty. Tables allow one to decide the significance of *r* for samples with degree-of-freedom defined as $df = n - 2$, to various probabilities.

LINEAR REGRESSION – least squares



Unexplained variation: $(Y-Y')^2$

Explained variation: $(Y'-\bar{Y})^2$

Coefficient of determination:

$$r^2 = \frac{\Sigma\left(Y'-\bar{Y}\right)^2}{\Sigma\left(Y-\bar{Y}\right)^2}$$

Correlation coefficient: r

r has the same sign as b

ERROR ANALYSIS IN YOUR REPORT

Typical Procedure:

- Experiments in lab will require you to estimate certain (physical) parameters
- Experiments need to be performed several times

Error Analysis in Report Should Include (all that applies):

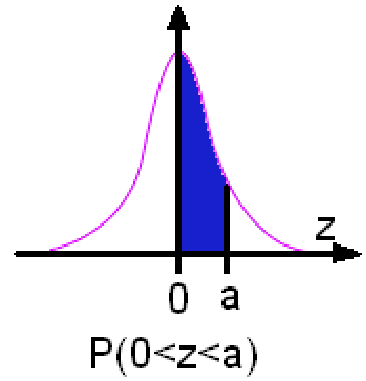
- Estimate of mean and standard deviation of parameter estimates

- Confidence interval of parameter estimate (based on t-distribution)

- Indication of error sources in experiments (what is causing errors?)
    - Intrinsic & Application Related Errors
    - Integration Errors

- Error Propagation
    - Especially if a parameters estimate is based on several measurements
    - Indicate how errors propagate (using Taylor series approximation)

Standard Normal Distribution Table

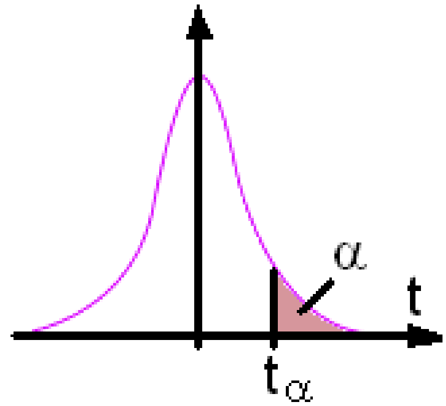


a	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

NOTE: Each entry of the table contains the value of $P(0<z<a)$, where rows = first decimal value of *a* and columns = second decimal value of *a*

Inverse T-Distribution Table

df	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.706	31.821	63.656	318.289	636.578
2	1.886	2.920	4.303	6.965	9.925	22.328	31.600
3	1.638	2.353	3.182	4.541	5.841	10.214	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.894	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.689
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.660
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
Infinity	1.282	1.645	1.960	2.326	2.576	3.091	3.291



NOTE: Each entry in the table contains the value of t_α, where rows = degrees of freedom and columns = $P(t > t_\alpha)$.

END OF LECTURE