Vibration Analysis Experiment: mode shapes and frequency response of a scaled flexible three-story building & helicopter propeller

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class information and lab handouts will be available on
http://maecourses.ucsd.edu/labcourse/

Main Objectives of Laboratory Experiment:

vibration analysis: mode shapes and frequency response

Ingredients:

- experiments with a shaker table/impact hammer
- application of vibration and dynamics theory
- learn to use a spectrum analyzer
- validation of experiments with dynamical model

Background Theory:

- Lagrange's method (separate handout posted on labcourse website http://maecourses.ucsd.edu/labcourse/)
- Ordinary Differential Equations (derivation & solutions)
- Linear System Theory (Laplace transform, Transfer function, Frequency Response, Eigenvalues/Eigenmodes)
- Fourier transform and spectral analysis

Outline of this lecture

- aim of experiment
- laboratory hardware
  - shaker table with flexible structure
  - helicopter blade
  - HP spectrum analyzer
- background theory
  - obtaining a model: Lagrange's method and FEM
  - mode shapes
  - transfer functions
  - frequency response estimation
- laboratory experiments
- what should be in your report

Aim of Lab Experiment

In this laboratory experiment we start with flexible structure (scaled three story building) and extend experiments to helicopter blade. Objective is to *understand and measure vibration models* and validate experimentally a Finite Element Model.

Aerodynamic vibration analysis is needed to

- reduce oscillation in flexible structures (fatigue and noise)
- understand mode shapes for lightweight construction

Aim of the experiment:

- insight in vibration analysis
- learn how to use a spectral analyzer
- experimental evaluation of Finite Element Model

Aim of Lab Experiment

See also NEES shaker table at UCSD http://nees.ucsd.edu/

Full scale shaker table for multi-story buildings.

We only have a small flexible structure in our lab. . .

Hardware in the Lab – 1st & 2nd week

shaker table and three story building with accelerometers

Hardware in the Lab – 3rd week

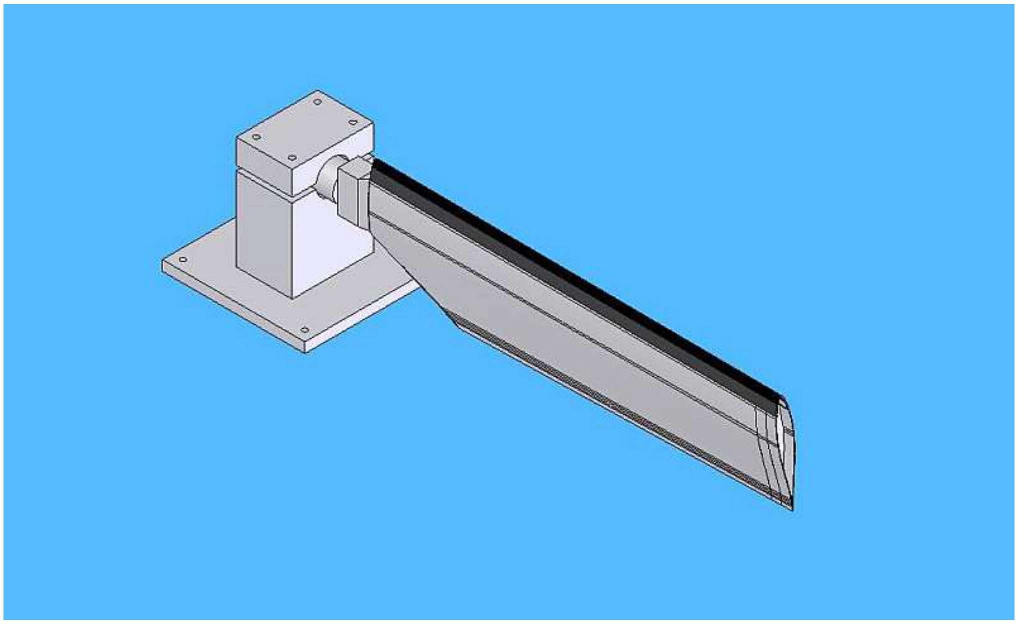


table-top mounted blade of helicopter tail rotor

Courtesy of Prof. J. Kosmatka, Dept. of Structural Engineering, UCSD

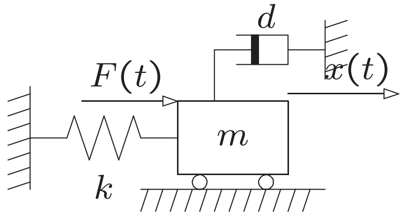
Hardware in the Lab – Spectrum Analyzer

Hewlett Packard HP 35670A Spectrum analyzer for data analysis

Background theory: obtaining a dynamic model

To study vibrations, with will use a dynamic model.

For example:



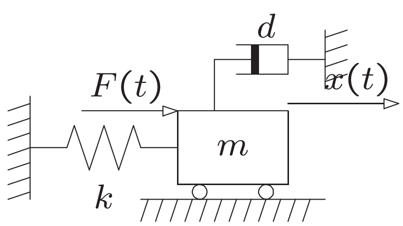
You (should) know: undamped resonance frequency:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

Relevant questions:

- Where does this come from or how is this derived?
- If this the resonance frequency, what is a resonance mode?
- How does this generalize to multiple masses (multiple degrees of freedom)?

Background theory: obtaining a dynamic model



Derived via equations of motion. Assume $d = 0$, no external force ($F = 0$), use 2nd Newton's law:

$$m\ddot{x}(t) + kx(t) = 0$$

Result: 2nd order ODE = dynamic model

Solutions that satisfy this ODE are of the from

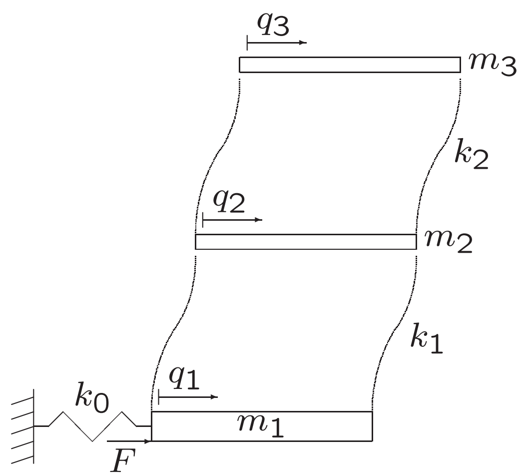
$$x(t) = C \sin(\omega_n t + \phi), \quad \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

and C, ϕ depend on initial conditions $x(0)$, $\dot{x}(0)$, but ω_n same.

Background theory: obtaining a dynamic model

What if we have multiple masses, each connect with springs?

Example: our three story building used in the lab experiments



Background theory: obtaining a dynamic model

Lagrange's equations offer a systematic way to formulate the equations of motion of a lumped mass system or a (flexible) system with multiple degrees of freedom.

Use of generalized coordinates: set of *independent* coordinates equal in number to the n degrees of freedom of the system under consideration

$$q_i, \; i = 1, 2, \ldots, n$$

Kinetic T and Potential U energy in generalized coordinates:

$$\begin{aligned} T &= T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \\ U &= U(q_1, \ldots, q_n) \end{aligned}$$

Background theory: Lagrange's method

Conservation of energy

$$d(T+U)=0$$

With $T(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n)$ and $U(q_1,\ldots,q_n)$ we have

$$dU \;:=\; \sum_{i=1}^{n}\frac{\partial}{\partial q_i}U(q_1,\ldots,q_n)dq_i \;=\sum_{i=1}^{n}\frac{\partial U}{\partial q_i}dq_i$$

and

$$\begin{aligned} dT \;:=&\; \sum_{i=1}^{n}\frac{\partial}{\partial q_i}T(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n)dq_i + \\ & \sum_{i=1}^{n}\frac{\partial}{\partial \dot{q}_i}T(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n)d\dot{q}_i \\ =& \;\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}dq_i + \sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}d\dot{q}_i \end{aligned}$$

Would be nice to remove second term with $d\dot{q}_i$ in dT

Background theory: Lagrange's method

Remove dependency of $d\dot{q}_i$ (the generalized velocity) in T via definition of kinetic energy via ($\frac{1}{2}\times$ mass $\times$ velocity^2):

$$T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}\dot{q}_i\dot{q}_j$$

so that

$$\frac{\partial T}{\partial \dot{q}_i}=\sum_{j=1}^{n}m_{ij}\dot{q}_j,\quad i=1,2,\ldots,n$$

making

$$T=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i$$

Immediately follows

$$2dT=\sum_{i=1}^{n}d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i+\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}d\dot{q}_i$$

Background theory: Lagrange's method

From $T(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n)$ we have:

$$dT = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i \qquad (1)$$

From $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j$ we have $T = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i$ and

$$2dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i \qquad (2)$$

Subtracting (1) from (2) removes dependency of $d\dot{q}_i$ (the generalized velocity) in T.

Background theory: Lagrange's method

Subtracting (1) from (2) yields

$$dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} dq_i = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \frac{\partial T}{\partial q_i} dq_i$$

Further simplification:

$$d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) dq_i$$

making

$$dT = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] dq_i$$

combining $d(T+U) = Q_i$ leads to Lagrange's equation for free body oscillation (no external forces):

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1,2,\ldots,n}$$

Background theory: Lagrange's method

Application of an external forces $F(t)$ will change the sum of potential U and kinetic energy T.

The change in energy can be quantified by the (virtual) work:

$$\delta W(t) = F(t)\delta q = \sum_{i=1}^{n} Q_i(t)\delta q_i$$

where $Q_i(t)$ denote the generalized forces in the generalized co-ordinate system $q_i,\ i = 1, 2, \ldots, n$

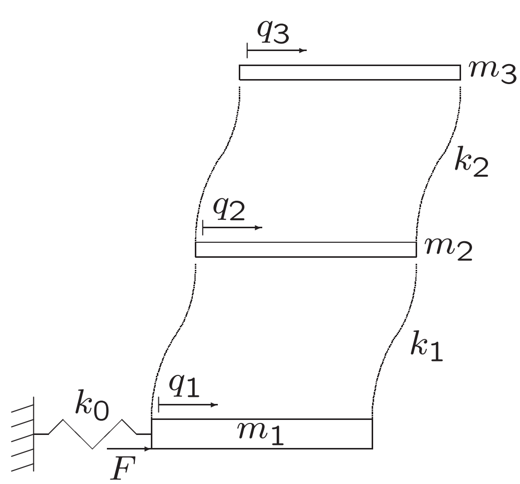
Combining $d(T + U) = Q_i$ leads to Lagrange's equation:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n}$$

where $Q_i =$ generalized forces found by virtual work.

Background theory: Lagrange's method applied to structure

Application: simple three-story building



The generalized coordinates $q_i,\ i = 1, 2, 3$ are chosen as the absolute horizontal position/displacement of the floors.

Background theory: Lagrange's method applied to structure

Kinetic energy T:

- Determined by the linear momentum p_i and velocity $\dot{q}_i$ of each floor.
- For each floor we have

$$T_i = \int p_i d\dot{q}_i$$

- With $p_i = m_i \dot{q}_i$ we see

$$T_i = \int p_i d\dot{q}_i = \int m_i \dot{q}_i d\dot{q}_i = \frac{1}{2} m_i \dot{q}_i^2$$

- Makes the total kinetic energy for the three story building:

$$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \frac{1}{2} m_3 \dot{q}_3^2$$

Background theory: Lagrange's method applied to structure

Potential energy U (without damping):

- Assuming linear (shear) stiffness k_i at each floor, U determined by spring force F_i^s and relative displacement $\bar{q}_i$.
- For each floor we have

$$U_i = \int F_i^s d\bar{q}_i$$

- With $F_i^s = k_i \bar{q}_i$ we see

$$U_i = \int F_i^s d\bar{q}_i = \int k_i \bar{q}_i d\bar{q}_i = \frac{1}{2} k_i \bar{q}_i^2$$

- This makes the total potential energy for the three story building (without damping):

$$U = \frac{1}{2} k_0 q_1^2 + \frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 (q_2 - q_3)^2$$

Background theory: Lagrange's method applied to structure

Potential energy U (with damping):

- Assuming linear stiffness k_i and linear (shear) damping d_i at each floor, U determined by spring force F_i^s, damping force F_i^d, relative displacement $\bar{q}_i$ and relative velocity $\dot{\bar{q}}_i$.
- For each floor we have

$$U_i = \int F_i^s d\bar{q}_i + \int F_i^d d\bar{q}_i$$

- With $F_i^s = k_i \bar{q}_i$ and $F_i^d = d_i \dot{\bar{q}}_i$ we see

$$U_i = \int k_i \bar{q}_i d\bar{q}_i + \int d_i \dot{\bar{q}}_i d\bar{q}_i = \frac{1}{2} k_i \bar{q}_i^2 + d_i \dot{\bar{q}}_i \bar{q}_i$$

- This makes the total potential energy for the three story building (with damping):

$$\begin{aligned} U \;=\; & \frac{1}{2}k_0 q_1^2 + \frac{1}{2}k_1(q_1 - q_2)^2 + \frac{1}{2}k_2(q_2 - q_3)^2 + \\ & d_0 \dot{q}_1 q_1 + d_1(\dot{q}_1 - \dot{q}_2)(q_1 - q_2) + d_2(\dot{q}_2 - \dot{q}_3)(q_2 - q_3) \end{aligned}$$

Background theory: Lagrange's method applied to structure

Summary for thee story building

Kinetic Energy:

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2$$

Potential Energy with damping:

$$\begin{aligned} U \;=\; & \frac{1}{2}k_0 q_1^2 + \frac{1}{2}k_1(q_1 - q_2)^2 + \frac{1}{2}k_2(q_2 - q_3)^2 + \\ & d_0 \dot{q}_1 q_1 + d_1(\dot{q}_1 - \dot{q}_2)(q_1 - q_2) + d_2(\dot{q}_2 - \dot{q}_3)(q_2 - q_3) \end{aligned}$$

In equilibrium we see that the total virtual work is given by

$$\delta W = F\delta q_1 \;\Rightarrow\; Q_1 = F,\; Q_2 = 0,\; Q_3 = 0$$

Background theory: Lagrange's method applied to structure

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

$$T = \tfrac{1}{2}m_1\dot{q}_1^2 + \tfrac{1}{2}m_2\dot{q}_2^2 + \tfrac{1}{2}m_3\dot{q}_3^2$$

$$U = \tfrac{1}{2}k_0q_1^2 + \tfrac{1}{2}k_1(q_1 - q_2)^2 + \tfrac{1}{2}k_2(q_2 - q_3)^2 + d_0\dot{q}_1q_1 + d_1(\dot{q}_1 - \dot{q}_2)(q_1 - q_2) + d_2(\dot{q}_2 - \dot{q}_3)(q_2 - q_3)$$

For $i = 1$:

$$\frac{\partial T}{\partial \dot{q}_1} = m_1\dot{q}_1 \ \Rightarrow \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1\ddot{q}_1$$

$$\frac{\partial T}{\partial q_1} = 0$$

$$\frac{\partial U}{\partial q_1} = (k_0 + k_1)q_1 - k_1q_2 + (d_0 + d_1)\dot{q}_1 - d_1\dot{q}_2$$

creating the first Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial U}{\partial q_1} = Q_1 = F$$

given by

$$m_1\ddot{q}_1 + (k_0 + k_1)q_1 - k_1q_2 + (d_0 + d_1)\dot{q}_1 - d_1\dot{q}_2 = F$$

Background theory: Lagrange's method applied to structure

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

$$T = \tfrac{1}{2}m_1\dot{q}_1^2 + \tfrac{1}{2}m_2\dot{q}_2^2 + \tfrac{1}{2}m_3\dot{q}_3^2$$

$$U = \tfrac{1}{2}k_0q_1^2 + \tfrac{1}{2}k_1(q_1 - q_2)^2 + \tfrac{1}{2}k_2(q_2 - q_3)^2 + d_0\dot{q}_1q_1 + d_1(\dot{q}_1 - \dot{q}_2)(q_1 - q_2) + d_2(\dot{q}_2 - \dot{q}_3)(q_2 - q_3)$$

For $i = 2$:

$$\frac{\partial T}{\partial \dot{q}_2} = m_2\dot{q}_2 \ \Rightarrow \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2\ddot{q}_2$$

$$\frac{\partial T}{\partial q_2} = 0$$

$$\frac{\partial U}{\partial q_2} = -k_1q_1 + (k_1 + k_2)q_2 - k_2q_3 - d_1\dot{q}_1 + (d_1 + d_2)\dot{q}_2 - d_2\dot{q}_3$$

creating the second Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial U}{\partial q_2} = Q_2 = 0$$

given by

$$m_2\ddot{q}_2 - k_1q_1 + (k_1 + k_2)q_2 - k_2q_3 - d_1\dot{q}_1 + (d_1 + d_2)\dot{q}_2 - d_2\dot{q}_3 = 0$$

Background theory: Lagrange's method applied to structure

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

$$T = \tfrac{1}{2}m_1\dot{q}_1^2 + \tfrac{1}{2}m_2\dot{q}_2^2 + \tfrac{1}{2}m_3\dot{q}_3^2$$

$$U = \tfrac{1}{2}k_0 q_1^2 + \tfrac{1}{2}k_1(q_1 - q_2)^2 + \tfrac{1}{2}k_2(q_2 - q_3)^2 +$$
$$d_0\dot{q}_1 q_1 + d_1(\dot{q}_1 - \dot{q}_2)(q_1 - q_2) + d_2(\dot{q}_2 - \dot{q}_3)(q_2 - q_3)$$

For $i = 3$:

$$\frac{\partial T}{\partial \dot{q}_3} = m_3\dot{q}_3 \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_3\ddot{q}_3$$

$$\frac{\partial T}{\partial q_3} = 0$$

$$\frac{\partial U}{\partial q_3} = -k_2 q_2 + k_2 q_3 - d_2\dot{q}_2 + d_2\dot{q}_3$$

creating the third and last Lagrange equation

$$m_3\ddot{q}_3 - k_2 q_2 + k_2 q_3 - d_2\dot{q}_2 + d_2\dot{q}_3 = 0$$

Background theory: mass, damping and stiffness matrices

The three Lagrange equations:

$$\begin{aligned}
m_1\ddot{q}_1 + (k_0 + k_1)q_1 - k_1 q_2 + (d_0 + d_1)\dot{q}_1 - d_1\dot{q}_2 &= F \\
m_2\ddot{q}_2 - k_1 q_1 + (k_1 + k_2)q_2 - k_2 q_3 - d_1\dot{q}_1 + (d_1 + d_2)\dot{q}_2 - d_2\dot{q}_3 &= 0 \\
m_3\ddot{q}_3 - k_2 q_2 + k_2 q_3 - d_2\dot{q}_2 + d_2\dot{q}_3 &= 0
\end{aligned}$$

Combined in matrix format:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\text{mass matrix } M} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} d_0 + d_1 & -d_1 & 0 \\ -d_1 & d_1 + d_2 & -d_2 \\ 0 & -d_2 & d_2 \end{bmatrix}}_{\text{damping matrix } D} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\text{stiffness matrix } K} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{Q} F$$

Background theory: mass, damping and stiffness matrices

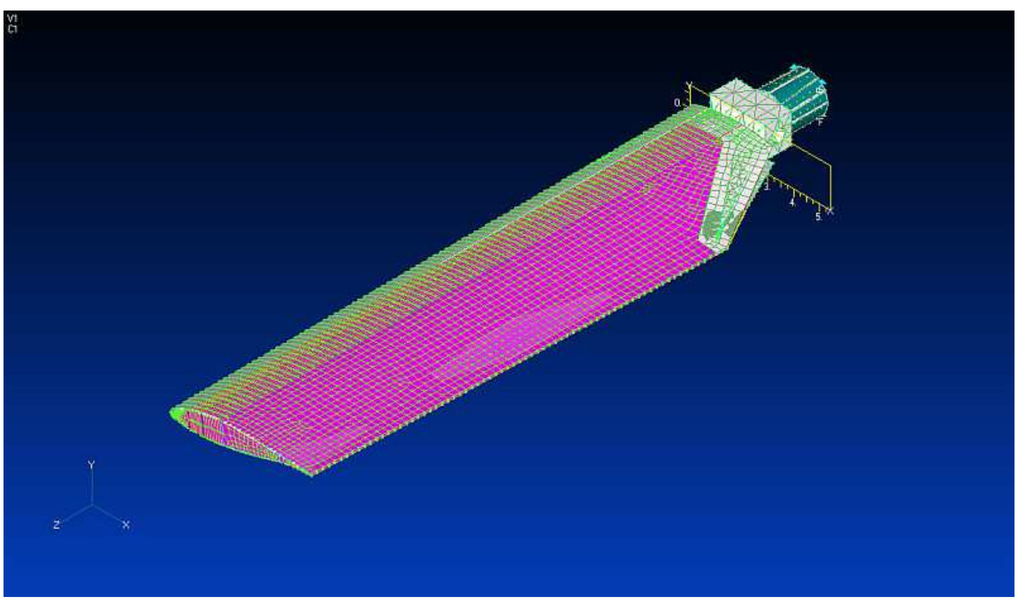
For many degrees of freedom, mass matrix M, stiffness matrix K and generalized force input matrix Q in

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = QF(t)$$

are computed via FEM (Finite Element Model)

- Create system of nodes via a mesh - density of mesh depends on configuration and expected stres
- Use mesh to program material and structural properties - standard elements in FEM model determine overall properties of meshed system (rod, beam, plate/shell/composite, shear)
- Specify boundary conditions (nodes restricted in motion and subjected to forces)

Background theory: mass, damping and stiffness matrices



meshing for blade of helicopter tail rotor
blade consists of skin and spar (separately meshed)
Courtesy of Prof. J. Kosmatka, Dept. of Structural Engineering, UCSD

Background theory: mode shapes

Consider (no damping to simplify formulae):

$$M\ddot{q}(t) + Kq(t) = Qu(t),\ \ M = M^T > 0,\ \ K = K^T \geq 0$$

there always exists a non-singular matrix P such that

$$P^T M P = I,\ \ P^T K P = \Omega^2 = \text{diagonal matrix}$$

Using $q(t) := Pp(t)$ we get

$$P^T[MP\ddot{p}(t) + KPp(t) = Qu(t)]\ \Rightarrow\ \ddot{p}(t) + \Omega^2 p(t) = \bar{Q}u(t)$$

P (and Ω^2) can be computed via generalized eigenvalue problem:

Computation of diagonal matrix $S = \Omega^2$ of *generalized eigenvalues* and a full matrix P whose columns are the corresponding eigenvectors so that

$$KP = MPS,\quad S = \Omega^2\ \ \text{diagonal}$$

Matlab implementation:

```
>> [P,S]=eig(K,M,'chol')
```


Background theory: mode shapes

$$P^T[MP\ddot{p}(t) + KPp(t) = Qu(t)]\ \Rightarrow\ \ddot{p}(t) + \Omega^2 p(t) = \bar{Q}u(t)$$

with

$$KP = MPS,\quad S = \Omega^2\ \ \text{diagonal}$$

Important observations:

- Due to $P^T M P = I$ and $P^T K P = \Omega^2 =$ diagonal matrix we get a set of *decoupled second order ODE's*
- Compare with our 2nd order ODE $m\ddot{x}(t) + kx(t) = F(t)$ we got from our simple mass/spring system earlier in our lecture

Background theory: mode shapes

Since Ω^2 is a diagonal matrix, we have a set of decoupled second order differential equations

$$\ddot{p}_i(t) + \omega_i^2 p_i(t) = \bar{q}_i u(t)$$

for which the homogeneous solution ($u(t) = 0$) is given by

$$p_i(t) = \sin(\omega_i t)$$

The diagonal elements ω_i of Ω contain the resonance frequencies of the mechanical or flexible structural system.

Eigenvalues leads to eigen modes by computing the generalized displacement q due to excitation

$$p_i(t) = \sin(\omega_i t)$$

Background theory: mode shapes

Consider set of n decoupled (homogeneous) equations

$$\ddot{p}(t) + \Omega^2 p(t) = 0$$

and consider normalized initial condition $p(0)$ on the jth element:

$$\dot{p}(0) = 0, \quad p(0) = \begin{bmatrix} p_1(0) \\ \vdots \\ p_n(0) \end{bmatrix} \text{ with } p_i(0) = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$$

will lead to dynamic response $p(t)$ in which *only* the jth element of $p(t)$ is non-zero (due to n decoupled equations).

Making

$$q_j = P\,p(0) = j\text{th column in } P$$

the jth eigenmode of the structure!

Background theory: mode shapes

Example of three story building: $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10,000$, $k1 = 1000$, $k_2 = 1000$,

$$M = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = 1000 \cdot \begin{bmatrix} 11 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and yielding

$$P \approx \begin{bmatrix} 0.0707 & -0.3035 & -0.0540 \\ 0.5347 & -0.0256 & 0.8446 \\ 0.8149 & 0.2802 & -0.5074 \end{bmatrix}$$

$$\Omega^2 \approx \begin{bmatrix} 343.81 & 0 & 0 \\ 0 & 1091.55 & 0 \\ 0 & 0 & 2664.64 \end{bmatrix}$$

computed via Matlab's `[P,S]=eig(K,M,'chol')`

Background theory: mode shapes

With example of three story building: $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10,000$, $k1 = 1000$, $k_2 = 1000$ we have

$$\Omega^2 \approx \begin{bmatrix} 343.81 & 0 & 0 \\ 0 & 1091.55 & 0 \\ 0 & 0 & 2664.64 \end{bmatrix}$$

and

1. First resonance mode at $\sqrt{343.81} \approx 18.54$ rad/s ≈ 2.85 Hz.
2. Second resonance mode at $\sqrt{1091.55} \approx 33.04$ rad/s ≈ 5.26 Hz.
3. Third resonance mode at $\sqrt{2664.64} \approx 51.62$ rad/s ≈ 8.22 Hz.

Note: these numbers are only valid for m_i, k_i, $i = 1, 2, 3$ mentioned above.

Background theory: mode shapes

With example of three story building: $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10,000$, $k1 = 1000$, $k_2 = 1000$ we have

$$P \approx \begin{bmatrix} 0.0707 & -0.3035 & -0.0540 \\ 0.5347 & -0.0256 & 0.8446 \\ 0.8149 & 0.2802 & -0.5074 \end{bmatrix}$$

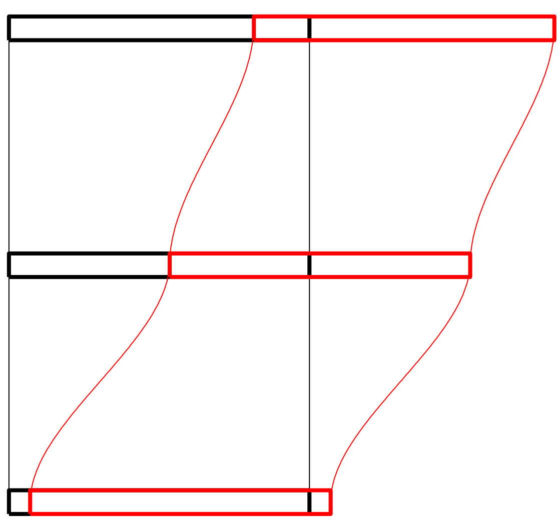
Hence: excitation with $u(t) = \sin(2\pi \cdot 2.85t)$ will predominantly exciting the 1st eigenmode

$$\bar{q}_1(t) = \begin{bmatrix} 0.707 & 0.5347 & 0.8149 \end{bmatrix}^T \sin(2\pi \cdot 2.85t)$$

so we have vibration with a (normalized) amplitude of floor 1: 0.0707, floor 2: 0.5347 and floor 3: 0.8149.

Indicates for 1st eigenmode that all floors move in same direction and displacement increases by floor.

Background theory: mode shapes



1st mode: ≈ 2.85 Hz with a (normalized) amplitude of floor 1: 0.0707, floor 2: 0.5347 and floor 3: 0.8149.

Background theory: mode shapes

With example of three story building: $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10,000$, $k1 = 1000$, $k_2 = 1000$ we have

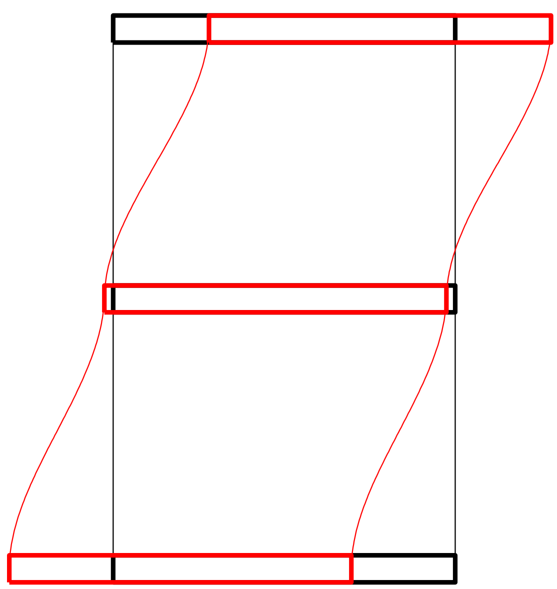
$$P \approx \begin{bmatrix} 0.0707 & -0.3035 & -0.0540 \\ 0.5347 & -0.0256 & 0.8446 \\ 0.8149 & 0.2802 & -0.5074 \end{bmatrix}$$

Excitation with $u(t) = \sin(2\pi \cdot 5.26t)$ will predominantly exciting the 2nd eigenmode.

So we have vibration with a (normalized) amplitude of floor 1: -0.3035, floor 2: -0.0256 and floor 3: 0.2802.

Indicates for 2nd eigenmode that floor 1 and floor 3 move in opposite direction, while floor 2 is hardly moving.

Background theory: mode shapes



2nd mode: ≈ 5.26 Hz with a (normalized) amplitude of floor 1: -0.3035, floor 2: -0.0256 and floor 3: 0.2802.

Background theory: mode shapes

With example of three story building: $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10,000$, $k1 = 1000$, $k_2 = 1000$ we have

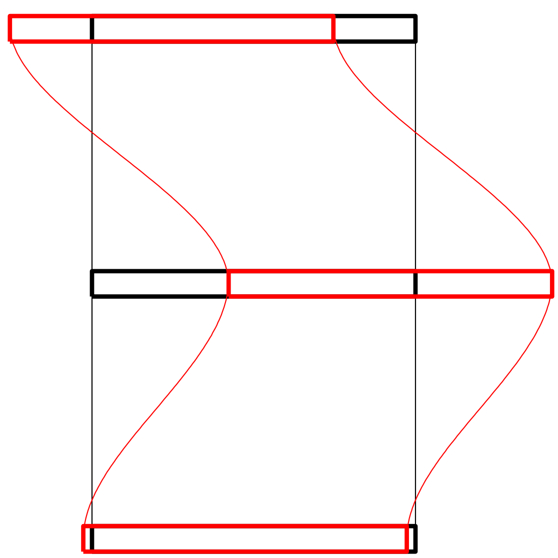
$$P \approx \begin{bmatrix} 0.0707 & -0.3035 & -0.0540 \\ 0.5347 & -0.0256 & 0.8446 \\ 0.8149 & 0.2802 & -0.5074 \end{bmatrix}$$

Excitation with $u(t) = \sin(2\pi \cdot 8.22t)$ will predominantly exciting the 3rd eigenmode.

So we have vibration with a (normalized) amplitude of floor 1: -0.0540, floor 2: -0.8446 and floor 3: -0.5074.

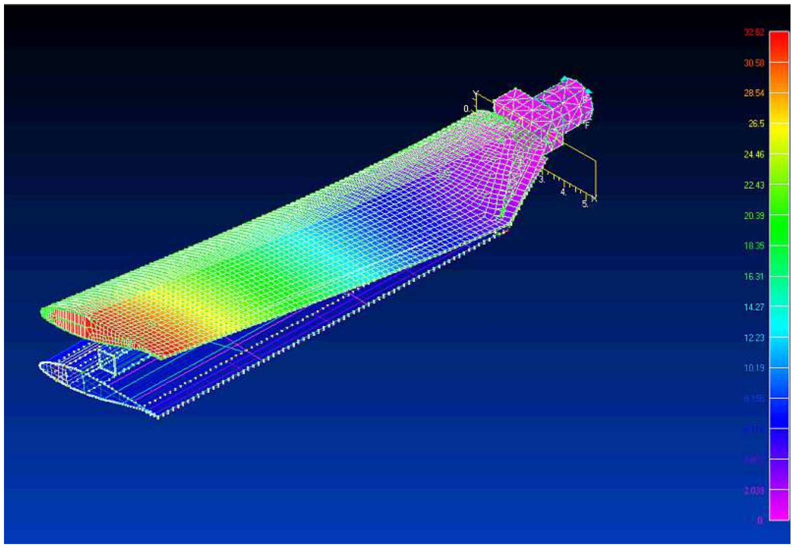
Indicates for 3rd eigenmode that floor 1 is hardly moving, while floor 2 and floor 3 move in opposite direction.

Background theory: mode shapes



3nd mode: ≈ 8.22 Hz with a (normalized) amplitude of floor 1: -0.0540, floor 2: -0.8446 and floor 3: -0.5074.

Background theory: mode shapes

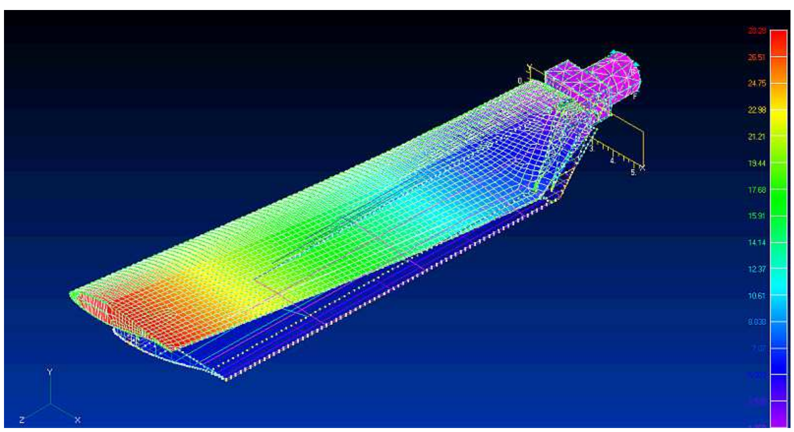


Helicopter blade - 1st mode: out-of-plane bending

See also http://maecourses.ucsd.edu/callafon/labcourse/movies/1st_mode_small.avi

Courtesy of Prof. J. Kosmatka, Dept. of Structural Engineering, UCSD

Background theory: mode shapes

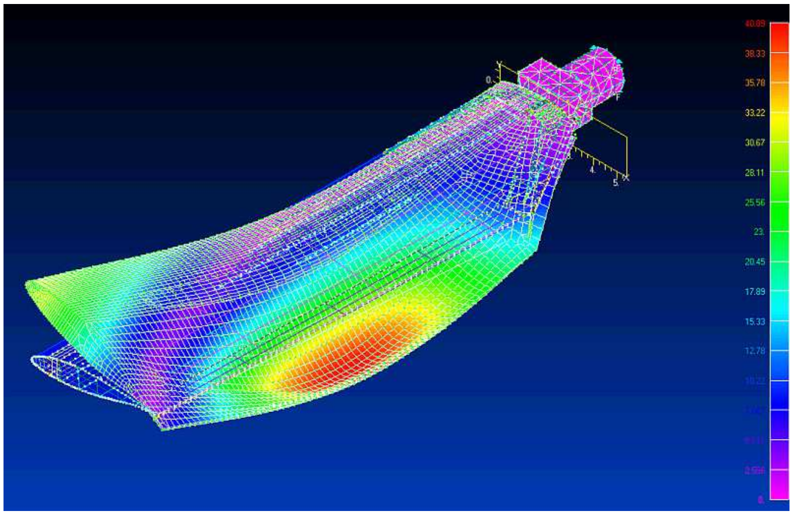


Helicopter blade - 2nd mode: 'in-plane' bending

http://maecourses.ucsd.edu/callafon/labcourse/movies/2nd_mode_in-plane_small.avi

Courtesy of Prof. J. Kosmatka, Dept. of Structural Engineering, UCSD

Background theory: mode shapes



Helicopter blade - 3rd mode: torsion mode

http://maecourses.ucsd.edu/callafon/labcourse/movies/3rd_mode_torsion_small.avi

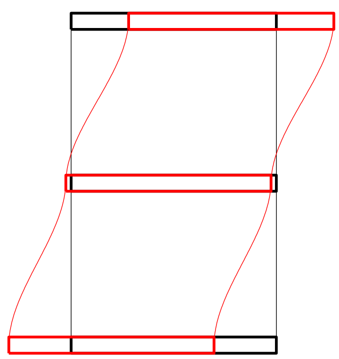
Courtesy of Prof. J. Kosmatka, Dept. of Structural Engineering, UCSD

Higher order modes, see: http://maecourses.ucsd.edu/callafon/labcourse/movies/

Background theory: transfer function

Next to resonance modes, zeros, anti-resonance modes or blocking properties are also important.

Example: building 2nd resonance mode – floor 2 was not moving!

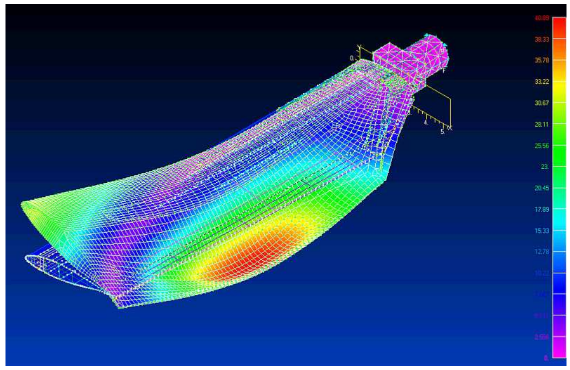


Relevant Questions:

- What will be transfer (function) from floor 1 to floor 2?
- What happens to this transfer function at the 2nd resonance frequency?

Background theory: transfer function

Example: blade 3rd resonance mode – large part of blade is not moving!



Relevant Questions:

- What will be transfer (function) from tip of blade to center of blade?
- What happens to this transfer function at the 3rd resonance frequency?

Background theory: transfer function

These questions can be answered (for small dimensional systems) by looking at the *transfer function* representation.

Recall: transfer function representation $G(s)$ and frequency response $G(j\omega)$:

- If $F(t)$ = input and $q(t)$ = output of linear ordinary differential equation, then Laplace domain yields

$$q(s) = G(s)F(s)$$

- Let $F(t) = \cos \omega t$ and $G(s)$ is stable.
  As $t \rightarrow \infty$, $q(t) = A(\omega)\cos(\omega t + \phi(\omega))$ where

$$\begin{aligned} A(\omega) &= |G(s)|_{s=j\omega} \\ \phi(\omega) &= \angle G(s)_{s=j\omega} \end{aligned}$$

Background theory: transfer function

Recall: mass matrix M, stiffness matrix K and generalized force input matrix Q are combined in the 2nd order differential equation.

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = QF(t)$$

Application of Laplace transform yields

$$[Ms^2 + Ds + K]q(s) = QF(s) \;\; \Rightarrow q(s) = G(s)F(s)$$

$$G(s) = [Ms^2 + Ds + K]^{-1}Q$$

where $G(s)$ is a 3×1 column vector transfer function for our three story building.

Background theory: transfer function

$$G(s) = [Ms^2 + Ds + K]^{-1}Q$$

Since $F(s)$ is scalar we can pick displacement of any floor $q_j(s)$ via:

$$q_j(s) = G_j(s)F(s)$$

where $G_j(s)$ is a scalar transfer function that models the dynamics between the input force F and the displacement of the jth floor.

You can now inspect the transfer function

- For a **Single Floor** (from Force $F(s)$ to displacement $q_j(s)$).
- **Between Floors** (from displacement $q_j(s)$ to displacement $q_i(s)$).

Background theory: transfer function

Looking at a single floor:

$$q_j(s) = G_j(s)F(s)$$

where

$$G_j(s) = j\text{th column of } G(s) = [Ms^2 + Ds + K]^{-1}Q$$

With

$$[Ms^2 + Ds + K]^{-1} = \frac{1}{\text{det}(Ms^2 + Ds + K)}\text{adj}(Ms^2 + Ds + K)$$

we see that

$$G_j(s) = \frac{\text{num}_j(s)}{\text{den}(s)}$$

where $\text{den}(s) = \text{det}[Ms^2 + Ds + K]$ is the same for all floors!

HENCE: One can compute the resonance frequencies (of all floors) by solving

$$\text{den}(s) = \text{det}(Ms^2 + Ds + K) = 0$$

Background theory: transfer function

Looking at between floors (you will have two accelerometers for measurements):

$$\begin{aligned} q_i(s) &= G_i(s)F(s),\ G_i(s) = \frac{\text{num}_i(s)}{\text{den}(s)} \\ q_j(s) &= G_j(s)F(s),\ G_j(s) = \frac{\text{num}_j(s)}{\text{den}(s)} \end{aligned}$$

allows you to look at the transfer function (the dynamics) between two floors:

$$\frac{q_i(s)}{q_j(s)} = \frac{G_i(s)F(s)}{G_j(s)F(s)} = \frac{G_i(s)}{G_j(s)}$$

making

$$q_i(s) = H_{ij}(s)q_j(s),\ H_{ij}(s) := \frac{G_i(s)}{G_j(s)} = \frac{\text{num}_i(s)}{\text{num}_j(s)}$$

NOTICE:

- $\text{den}(s)$ drops out
- resonance modes in $H_{ij}(s)$ are determined by $\text{num}_j(s) = 0$

Background theory: transfer function

MAIN RESULT: for a three story building *without damping*:

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

can make

$$\begin{aligned} \text{num}_i(s) &= C_i(s^2+\omega_1^2)(s^2+\omega_2^2) \quad \text{or} \\ \text{num}_i(s) &= C_i(s^2+\omega_1^2) \quad \text{or} \\ \text{num}_i(s) &= C_i \end{aligned}$$

where $\omega_{1,2} =$ 'anti' resonance frequency, $C_i =$ constant (gain).

With

$$q_i(s) = H_{ij}(s) q_j(s), \quad H_{ij}(s) := \frac{G_i(s)}{G_j(s)} = \frac{\text{num}_i(s)}{\text{num}_j(s)}$$

we now have:

- 'anti-resonance modes' determined by $\text{num}_i(s) = 0$
- resonance modes determined by $\text{num}_j(s) = 0$

Background theory: transfer function

$$H_{ij}(s) = \frac{\text{num}_i(s)}{\text{num}_j(s)}$$

- Implication of 'resonance modes': if $\text{num}_j(s)$ satisifies

  $$\text{num}_j(s) = C_j(s^2+\omega_1^2)(s^2+\omega_2^2)$$

  then

  $$|H_{ij}(s)| = \infty \text{ for } s = j\omega_1 \text{ and } s = j\omega_2$$

  Hence: sinusoid excitation with frequency ω_1 or ω_2 rad/s creates infinitely large displacement.
- Implication of 'anti-resonance modes': if $\text{num}_i(s)$ is

  $$\text{num}_i(s) = C_i(s^2+\omega_3^2)$$

  then

  $$|H_{ij}(s)| = 0 \text{ for } s = j\omega_3$$

  Hence: sinusoid excitation with frequency ω_3 rad/s creates zero displacement.

Background theory: modeling (without damping)

Modeling without damping:

The transfer function $H_{ij}(s)$ from accelerometer $\ddot{q}_j(s)$ at floor j to accelerometer $\ddot{q}_i(s)$ at floor i is given by the general form

$$q_i(s) = H_{ij}(s) q_j(s), \quad H_{ij}(s) := \frac{G_i(s)}{G_j(s)} = \frac{\text{num}_i(s)}{\text{num}_j(s)}$$

where (without damping) $H_{ij}(s)$ is given by

$$H_{ij}(s) = C_i \cdot \frac{(s^2 + \omega_1^2)}{(s^2 + \omega_2^2)(s^2 + \omega_3^2)}$$

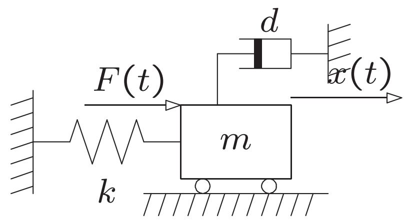
where

$$C_i = \frac{\omega_2^2 \omega_3^2}{\omega_1^2} = \text{scaling or gain}$$

$$\omega_i = \text{frequencies of undamped (anti) resonances}$$

Background theory: modeling (with damping)

Recall transfer function $G(s)$ of *single* mass m, damper d and stiffness k system:



Laplace transform of $m\ddot{x}(t) = F(t) - kx(t) - d\dot{x}(t)$:

$$ms^2 x(s) + dsx(s) + kx(s) = F(s), \quad \Rightarrow \quad x(s) = \underbrace{\frac{1}{ms^2 + ds + k}}_{G(s)} F(s),$$

The transfer function $G(s)$ written as standard 2nd order system:

$$G(s) = \frac{1}{ms^2 + ds + k} = \frac{1}{k} \cdot \frac{\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}$$

with $\omega_n := \sqrt{\frac{k}{m}}$ (**resonance**) and $\beta := \frac{1}{2}\frac{d}{\sqrt{mk}}$ (**damping ratio**)

Background theory: modeling (with damping)

Modeling with damping:

The transfer function $H_{ij}(s)$ from accelerometer $\ddot{q}_j(s)$ at floor j to accelerometer $\ddot{q}_i(s)$ at floor i is given by the general form

$$q_i(s) = H_{ij}(s)q_j(s), \ \ H_{ij}(s) := \frac{G_i(s)}{G_j(s)} = \frac{\text{num}_i(s)}{\text{num}_j(s)}$$

where (with damping) $H_{ij}(s)$ is given by

$$H_{ij}(s) = C_i \cdot \frac{(s^2 + 2\beta_1\omega_1 s + \omega_1^2)}{(s^2 + 2\beta_2\omega_2 s + \omega_2^2)(s^2 + 2\beta_3\omega_3 s + \omega_3^2)}$$

where

$$\begin{aligned} C_i = \frac{\omega_2^2\omega_3^2}{\omega_1^2} &= \text{scaling or gain} \\ \omega_i &= \text{frequencies of undamped (anti) resonances} \\ \beta_i &= \text{damping ratio of (anti) resonances} \end{aligned}$$

Background theory: modeling (example)

Example:

$$\omega_1 = 2\pi \cdot 15, \ \omega_2 = 2\pi \cdot 8, \ \omega_3 = 2\pi \cdot 25,$$
$$\beta_1 = 0.01, \ \beta_2 = 0.01, \ \beta_3 = 0.01, \text{ and } K = 1$$

results in a model

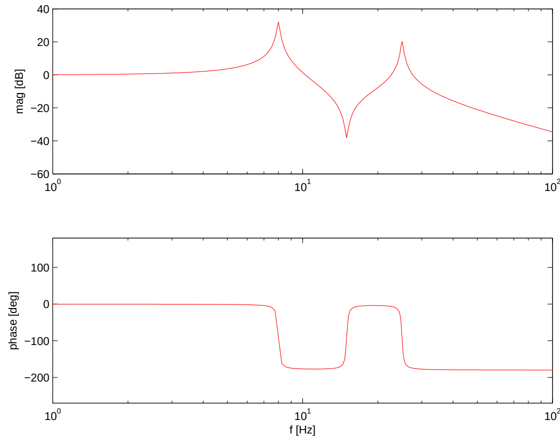
$$H_{ij}(s) = \frac{7018s^2 + 1.323 \cdot 10^4 s + 6.234 \cdot 10^7}{s^4 + 4.147s^3 + 2.72 \cdot 10^4 s^2 + 3.274 \cdot 10^4 s + 6.234 \cdot 10^7}$$

Matlab commands:

```
w2=2*pi*8;w1=2*pi*15;w3=2*pi*25;
beta1=0.01;beta2=0.01;beta3=0.01;K=1;
num=[1 2*beta1*w1 w1^2];                                  (3)
den=conv([1 2*beta2*w2 w2^2],[1 2*beta1*w3 w3^2]);
Hij=K*w2^2*w3^2/w1^2*tf(num,den);
```

Background theory: modeling (example)

Results in a Bode plot (what does this mean?)



Matlab commands (3) and:

```
myf=logspace(0,2,500);
[m,p]=bode(Hij,2*pi*myf);
subplot(2,1,1),semilogx(myf,20*log10(abs(squeeze(m)))),
subplot(2,1,2),semilogx(myf,squeeze(p))
```
(4)

Background theory: frequency response estimation

What is the best way to see sinusoids being amplified (resonance) or being blocked (anti resonance) in a signal $\ddot{q}_j(t)$?

Compute the Fourier transform of the signal $\ddot{q}_j(t)$

$$Q_N(\omega_n) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \ddot{q}(k\Delta_T)e^{-i\omega_n k\Delta T},\ \omega_n = n \cdot \frac{2\pi}{N\Delta_T}$$

that writes $\ddot{q}_j(t)$ as a sum of $N/2$ sinusoids

$$e^{-i\omega_n k\Delta T} = \cos(\omega_n k\Delta T) - i\sin(\omega_n k\Delta T)$$

Simply look at the spectrum of the signal $\ddot{q}_j(t)$:

$$|Q_N(\omega_n)|^2 \text{ over } \omega_n = n \cdot \frac{2\pi}{N\Delta_T},\ n = 0, 1, \ldots, N/2$$

also know as the *periodogram* and can be estimated by the Spectrum analyzer in the lab.

Background theory: frequency response estimation

Spectrum analyzer samples signals $q(k\Delta T),\ k = 1, 2, \ldots, N$ and computes Discrete Fourier Transform (DFT) over N time samples

$$Q_N(\omega) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} q(k\Delta T) e^{-i\omega k \Delta T}$$

MAIN RESULT:

Let two sampled signals u and y be related by a transfer function G, then

$$Y_N(\omega) = G(i\omega)U_N(\omega) + V_N(\omega) + R_N(\omega)$$

where $Y_N(\omega)$ and $U_N(\omega)$ are the DFT of $y(k\Delta T)$ and $u(k\Delta T)$, $V_N(\omega)$ is the DFT of possible noise on the measurements and $R_N(\omega)$ is due to the effect of (unknown) initial conditions.

Background theory: frequency response estimation

The DFT Y_N and U_N in

$$Y_N(\omega) = G(i\omega)U_N(\omega) + V_N(\omega) + R_N(\omega)$$

can be used to estimate the frequency response of $G(s)$:

$$\widehat{G}(i\omega) := \frac{Y_N(\omega)}{U_N(\omega)} = G(i\omega) + \frac{V_N(\omega)}{U_N(\omega)} + \frac{R_N(\omega)}{U_N(\omega)}$$

NOTE: $\widehat{G}(i\omega) = G(i\omega)$ if effect of $V_N(\omega)$ and $R_N(\omega)$ can be eliminated.

Effect of $V_N(\omega)$ and $R_N(\omega)$ is eliminated by spectral analysis:
(1) performing many estimates and averaging
(2) use of periodic input signals or averaging of initial conditions

$$\text{Resulting estimate}: \ \widehat{G}(i\omega) = \frac{\widehat{\Phi}_{yu}(\omega)}{\widehat{\Phi}_{uu}(\omega)}$$

where $\widehat{\Phi}_{yu}(\omega)$ and $\widehat{\Phi}_{uu}(\omega)$ are spectral estimates (averaged Fourier estimates)

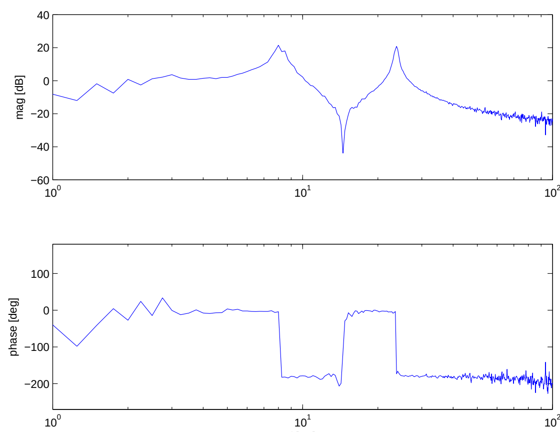
Background theory: frequency response estimation

$$\text{Estimate}: \; \widehat{G}(i\omega) = \frac{\widehat{\Phi}_{yu}(\omega)}{\widehat{\Phi}_{uu}(\omega)}$$

computed via $y(t)$ = Channel 2, $u(t)$ = Channel 1.

Background theory: frequency response estimation

Typical response (from floor 1 to floor 2)



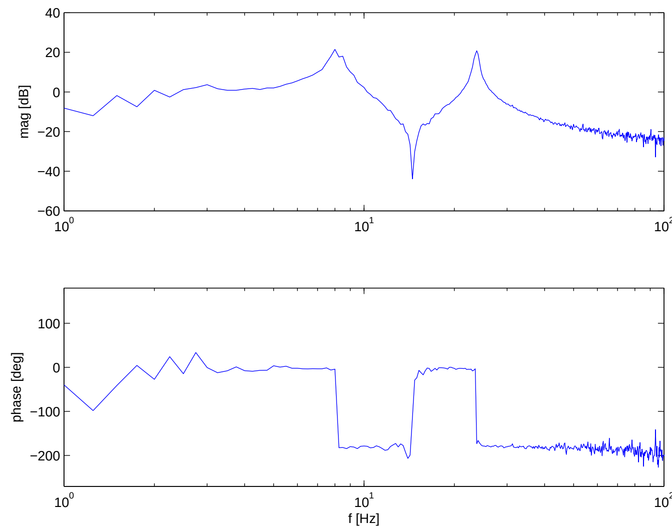
Matlab commands (see also `help gettrace`)

```
load mydata % created via [G,f]=gettrace(1); save mydata G f
subplot(2,1,1),semilogx(f,20*log10(abs(G)))
subplot(2,1,2),semilogx(f,180/pi*unwrap(angle(G)))
```

(5)

Background theory: frequency response estimation

Typical response (from floor 1 to floor 2)



Notice: 1st resonance frequency $\hat{f}_1$ around 8Hz, 3rd resonance frequency $\hat{f}_3$ around 25Hz and the 2nd resonance frequency $\hat{f}_2$ around 15Hz that makes the floor 2 'stands still' (anti-resonance)

Background theory: parameter estimation

Recall:

$$\ddot{q}_i(s) = H_{ij}(s)\ddot{q}_j(s),\ H_{ij}(s) = \frac{G_i(s)}{G_j(s)}$$

and typically (with damping in structure),

$$H_{ij}(s) = C_i \cdot \frac{(s^2 + 2\beta_1\omega_1 s + \omega_1^2)}{(s^2 + 2\beta_2\omega_2 s + \omega_2^2)(s^2 + 2\beta_3\omega_3 s + \omega_3^2)}$$

where

$C_i = \frac{\omega_2^2\omega_3^2}{\omega_1^2}$ = scaling or gain

ω_k = (anti) resonance frequency [rad/s] for $k = 1, 2, 3$

β_k = damping ratio $[0 \cdots 1]$ for $k = 1, 2, 3$

HENCE: you can estimate the above parameters from the frequency response measurements to obtain a model.

Requires estimation of ω_k and β_k.

Background theory: parameter estimation

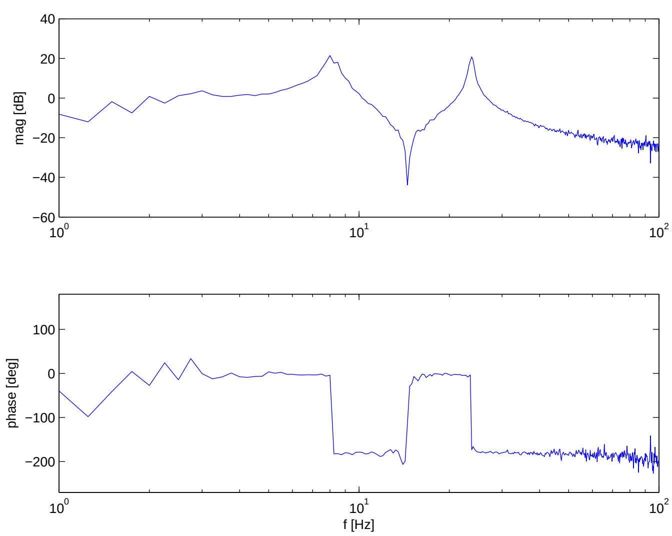
With

$$H_{ij}(s) = C_i \cdot \frac{(s^2 + 2\beta_1\omega_1 s + \omega_1^2)}{(s^2 + 2\beta_2\omega_2 s + \omega_2^2)(s^2 + 2\beta_3\omega_3 s + \omega_3^2)}$$

Frequency response is obtained when substituting $s = j\omega$ and you can see:

- $|H_{ij}(j\omega)|_{\omega=0} = 1$, so 1 is DC-gain.
- $|H_{ij}(j\omega)|_{\omega=\omega_1} =$ small, so ω_1 refers to blocking zero or anti-resonance frequency observed in floor 2.
- $|H_{ij}(j\omega)|_{\omega=\omega_2,\omega_3} =$ large, so ω_2 and ω_3 resonance frequencies.

Background theory: parameter estimation

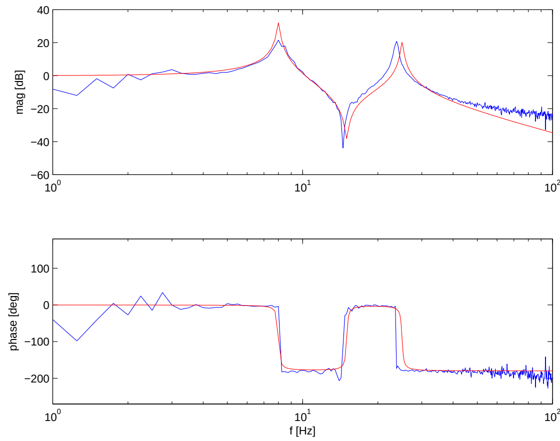


From measured frequency response, estimate model parameters:

$$\begin{aligned} \omega_k &= \text{(anti) resonance frequency [rad/s] for } k = 1,2,3 \\ \beta_k &= \text{damping ratio } [0 \cdot 1] \text{ for } k = 1,2,3 \end{aligned}$$

Background theory: parameter estimation

Compare measured and modeled frequency response:



Matlab commands (3), (4), (5) and below:

```
subplot(2,1,1),semilogx(f,20*log10(abs(G)),myf,20*log10(abs(squeeze(m))))
subplot(2,1,2),semilogx(f,180/pi*unwrap(angle(G)),myf,squeeze(p))
```

(6)

Background for Lab Work: week 1

Week 1 experiments: building resonance mode and resonance frequency estimation via sinusoidal experiments

- Learn use spectrum analyzer to create and measure signals.
- Excite structure with sinusoidal input using shaker table.
- Estimate resonance frequencies $\widehat{\omega}_k = 2\pi \widehat{f}_k$ by *visual inspection of resonance mode shapes*.
- Characterize mode shape at those resonance frequencies $\widehat{f}_k$ measuring by the (normalized/relative) size of oscillation of each floor $\ddot{q}_i(t)$ using accelerometers.
- Perform experiments several time for statistical analysis on estimates $\widehat{f}_k$.
- Measure acceleration signals $\ddot{q}_i(t)$ for all floors.

Background for Lab Work: week 2

Week 2 experiments: building resonance frequency ω_k and damping β_k estimation via frequency response estimation

- Use spectrum analyzer to measure frequency responses $\widehat{G}(i\omega)$ between different floors.
- Excitaton with swept sine $u(t) = \sin \omega(t)t$ or random signals $\mathrm{E}\{u(t)\} = 0$, $\mathrm{E}\{u(t)^2\} = \lambda$.
- Re-estimate resonance (and anti-resonance) frequencies ω_k and damping ratios β_k based on frequency response estimation.
- Perform experiments several time for statistical analysis on estimates ω_k and β_k.
- Create a model $H_{21}(s)$ (from floor 1 to floor 2) and validate frequency response of model $H_{21}(j\omega)$ with measured frequency response $\widehat{G}(i\omega)$.

Laboratory Work: week 3

Week 3 experiments: helicopter blade resonance frequency ω_k and damping β_k estimation via frequency response estimation

- Mount helicopter blade for experiments, place two accelerometers at strategic locations (use mode shapes from FEM analysis). Keep track of location used for experiments.
- Excitaton with swept sine $u(t) = \sin \omega(t)t$ or random signals $\mathrm{E}\{u(t)\} = 0$, $\mathrm{E}\{u(t)^2\} = \lambda$.
- Use spectrum analyzer to measure frequency responses $\widehat{G}(i\omega)$ between accelerometers.
- Estimate resonance frequencies ω_k and damping β_k of 1st, 2nd and 3rd resonance modes.
- Perform experiments several time for statistical analysis on estimates ω_k.

What should be in your report (1-2)

- Abstract
  Standalone - make sure it contains clear statements w.r.t motivation, purpose of experiment, main findings (numerical) and conclusions.
- Introduction
  - Motivation (why are you doing this experiment)
  - Short description of the main engineering discipline (vibration)
  - Answer the question: what is the aim of this experiment/report?
- Theory
  - Summary of Lagrange's method
  - Dynamic model for three story bulding
  - Modeling & transfer functions
  - Parameter estimation

What should be in your report (2-2)

- Experimental Procedure
  - Short description of experiment
  - How are experiments done (detailed enough so someone else could repeat them)
- Results
  - Measured acceleration and mode shapes for building
  - Parameter estimation for building
  - Model validation (estimated and modeled freq. response)
  - Parameter estimation for helicopter blade
- Discussion
  - Why are simulation results different from experiments?
  - Could the model be validated?
- Conclusions
- Error Analysis
  - Mean, standard deviation and 99% confidence intervals of estimated parameters ω_k, β_k from data
  - How do errors propagate?