Lagrange's Method

application to the vibration analysis of a flexible structure*

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Abstract

This handout gives a short overview of the formulation of the equations of motion for a flexible system using Lagrange's equations. Lagrange's equations provides an analytic method to analyze dynamical systems by a scalar procedure starting from the scalar quantities of kinetic energy, potential energy and (virtual) work, expressed in terms of generalized coordinates.

Keywords: Dynamical Systems, Kinetic Energy, Potential Energy, Generalized Coordinates, Virtual Work, Lagrange

Introduction

Lagrange's equations offer a systematic way to formulate the equations of motion of a mechanical system or a (flexible) structural system with multiple degrees of freedom. A scalar approach is obtained by expressing the scalar quantities of kinetic and potential energy in terms of generalized coordinates. The treatment of Lagrange's method is only brief in this document, but provides enough background information for the vibration analysis of the flexible structure used in the MAE laboratory course.

Potential and Kinetic Energy in Generalized Coordinates

Considering a system with n degrees of freedom, generalized coordinates refer to any set of *independent* coordinates equal in number to the n degrees of freedom of the system under consideration. In this document, the generalized coordinates are denoted by $q_i,\ i = 1, 2, \ldots, n$ and are used to express the scalar notion of kinetic energy T and potential energy U.

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Potential energy U of a mechanical or flexible structural system typically only depends on the position of the system. Kinetic energy T typically depends on velocity, but may be also be position dependent. In terms of generalized coordinates $q_i,\ i = 1, 2, \ldots, n$, the scalar notion of kinetic energy T and potential energy U can be expressed as functions

$$\begin{array}{rcl} T & = & T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \\ U & = & U(q_1, \ldots, q_n) \end{array} \tag{1}$$

that depend on the generalized positions q_i and generalized velocity $\dot{q}_i$ for $i = 1, 2, \ldots, n$.

Derivation of Lagrange's Equations

Considering an conservative system, where all external and internal forces have a potential. In that case, the sum of kinetic energy T and potential energy U will be constant and the differential is equal to zero:

$$d(T + U) = 0 \tag{2}$$

The above equation is basically a statement of the principle of conservation of energy. With the kinetic energy T and the potential energy U written as in (1), Lagrange's equations can be derived by summing up the kinetic and potential energy over all *generalized coordinates* $q_i,\ i = 1, 2, \ldots, n$.

With T and U given in (1) it is easy to see that

$$dU := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} U(q_1, \ldots, q_n) dq_i \tag{3}$$

and

$$dT := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) dq_i + \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i} T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) d\dot{q}_i \tag{4}$$

In the remainder of the derivation, the arguments q_i and $\dot{q}_i$ of $U(\cdot)$ and $T(\cdot)$ are dropped for brevity.

The second term in dT depends on perturbations $d\dot{q}_i$ (the generalized velocity) and can be eliminated by considering the equation for kinetic energy ($\frac{1}{2}mv^2$) in generalized coordinates

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \tag{5}$$

where m_{ij} denote the coefficients of the mass matrix in generalized coordinates. The coefficients of such a *generalized mass matrix* are presented in (13) and a

discussed is deferred to later in this document. For now it suffices to know that $m_{ij} = m_{ji}$ and differentiation of T with respect to $\dot{q}_i$ gives

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij}\dot{q}_j, \quad i = 1, 2, \ldots, n$$

The result can be back substituted into the expression for the kinetic energy T in (5) to obtain

$$T = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i$$

The second term with $d\dot{q}_i$ can be eliminated from (4) using the product rule:

$$2dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i$$

and subtraction of (4) from the above equation yields

$$dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} dq_i$$

Further simplification of this expression is obtained by the fact that

$$d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) dq_i$$

making

$$dT = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] dq_i \tag{6}$$

With (3) and (6), the equation of conservation of energy (2) now becomes

$$d(T + U) = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right] dq_i = 0$$

Since q_i denote the generalized coordinates that are a set of *independent* coordinates, the above expression is satisfied if and only if

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n \tag{7}$$

Equation (7) constitutes Lagrange's equation for a conservative system, where all external and internal forces have a potential. For systems that are non-conservative, Lagrange's equation in (7) can be generalized by including a non-zero right side term

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n \tag{8}$$

where Q_i denotes the (generalized) forces.

It is clear that writing down Lagrange's equations requires the partial derivative of the scalar functions of the kinetic energy $T(q_i, \dot{q}_i)$ and potential energy $U(q_i)$ with respect to the generalized coordinates q_i and generalized velocity $\dot{q}_i$ for each $i = 1, 2, \ldots, n$. A short-hand version of Lagrange's equations in (7) and (8) can obtained by defining a single scalar Lagrange function

$$L(q_i, \dot{q}_i) := T(q_i, \dot{q}_i) - U(q_i) \tag{9}$$

and realizing that

$$\frac{\partial}{\partial \dot{q}_i} U(q_i) = 0$$

As a result, (8) can also be written as the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n$$

where L is the Lagrangian defined in (9).

Application of Lagrange's method

To find the equations of motion for a flexible or mechanical system using Lagrange's equations boils down to the following steps:

1. *Definition of the generalized coordinates q_i*
   This can be any set of *independent* coordinates equal in number to the n degrees of freedom of the system under consideration. Representing an n-degree of freedom system by n particles, the position r_j of the jth particle can then be expressed in terms of the generalized coordinates

   $$r_j = r_j(q_1, \ldots, q_n) \tag{10}$$

   where r_j denotes a vector variable.

2. *Formulation of the kinetic energy T*
   With the position r_j given in (10) in terms of the generalized coordinates, the velocity of the jth particle is given by

   $$v_j := \frac{d}{dt} r_j = \dot{r}_j = \sum_{i=1}^{n} \frac{\partial r_j}{\partial q_i} \dot{q}_i \tag{11}$$

   in terms of the generalized coordinates q and generalized velocity $\dot{q}$. The total kinetic energy T of the system is given by

   $$T := \frac{1}{2} \sum_{k=1}^{n} m_k v_k \cdot v_k \tag{12}$$

   which is expressed as an inner product of the velocity v_k times the mass m_k for each particle $k = 1, 2, \ldots, n$. For the Lagrange equation, the kinetic

energy in terms of the generalized coordinates q and generalized velocity $\dot{q}$ is required. With (11) the kinetic energy T can be written as

$$\begin{aligned} T = \frac{1}{2}\sum_{k=1}^{n} m_k v_k \cdot v_k &= \frac{1}{2}\sum_{k=1}^{n} m_k \left(\sum_{i=1}^{n} \frac{\partial r_k}{\partial q_i}\dot{q}_i\right) \cdot \left(\sum_{j=1}^{n} \frac{\partial r_k}{\partial q_j}\dot{q}_j\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n} m_k \frac{\partial r_k}{\partial q_j}\cdot\frac{\partial r_k}{\partial q_i}\right)\dot{q}_i\dot{q}_j \end{aligned}$$

A short-hand notation for the kinetic energy in terms of the generalized coordinates can be obtained by the definition of the *generalized mass matrix* M. In case the generalized mass matrix M has coefficients m_{ij} given by

$$m_{ij} := \sum_{k=1}^{n} m_k \frac{\partial r_k}{\partial q_j}\cdot\frac{\partial r_k}{\partial q_i} \tag{13}$$

where r_j is given in (10). As a result, the kinetic energy T can be written as

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{q}^T M \dot{q} \tag{14}$$

which has a similar format as (12) but is now expressed in terms of the generalized velocity $\dot{q}$ and the generalized mass matrix M with the coefficients m_{ij} given in (13).

3. *Formulation of the potential energy U*
In a conservative system, the forces that have a potential can be derived from the potential energy U. Furthermore, the potential energy is a function of the generalized coordinates q. Expanding $U(q_1, \ldots, q_n)$ in a Taylor series expansion around the equilibrium position $q = q_0$ yields

$$U(q_1,\ldots,q_n) = U_0 + \sum_{i=1}^{n} q_i \left.\frac{\partial U}{\partial q_i}\right|_{q=q_0} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q=q_0} + \cdots$$

where $U_0 = U(q_0)$ is the (ground) potential energy at $q = q_0$ that can be set to zero. Since $q = q_0$ is assumed to be the equilibrium condition, we also have

$$\left.\frac{\partial U}{\partial q_i}\right|_{q=q_0} = 0$$

which leaves

$$U(q_1,\ldots,q_n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q=q_0} + \text{ higher order terms}$$

Ignoring the higher order terms in small oscillations around the equilibrium position allows the potential energy U to be written as

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q=q_0}$$

which can be reduced to the short-hand notation

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} q_i q_j = \frac{1}{2} q^T K q \tag{15}$$

where k_{ij} are the coefficients

$$k_{ij} := \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{q=q_0} \tag{16}$$

of the *generalized stiffness matrix* K. The potential energy of a spring with a stiffness K is similar to the short-hand notation of the potential energy U in (15).

4. *Formulation of the generalized forces* Q_i
In case of non-potential forces, the right hand side of the Lagrange's equations contains a constant term Q_i, denoting the generalized work of the generalized forces F_i in the generalized coordinates q_i. When the system is in equilibrium, the total virtual work δW can be found by summing the work δW_j with an inner product of the small force F_j and a resulting perturbation δr_j of the position r_j of particles

$$\delta W := \sum_{j=1}^{n} \delta W_j = \sum_{j=1}^{n} F_j \cdot \delta r_j$$

With (10) the perturbation δr_j can be written in terms of the generalized coordinates

$$\delta r_j = \sum_{i=1}^{n} \frac{\delta r_j}{\delta q_i} \delta q_i$$

making

$$\delta W = \sum_{j=1}^{n} sum_{i=1}^{n} F_j \cdot \frac{\delta r_j}{\delta q_i} \delta q_i$$

With the definition of the *generalized forces* Q_i given by

$$Q_i := \sum_{j=1}^{n} F_j \cdot \frac{\delta r_j}{\delta q_i} \tag{17}$$

the virtual work δW of the system can be written as

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i$$

and the generalized force Q_i is used for each Lagrange equation $i, = 1, \ldots, n$ to take into account the virtual work for each generalized coordinate.

All information to write down the n Lagrange's equations is available when the above steps are completed. This results in n equations in n (independent) generalized coordinates $q_i,\ i = 1, \ldots, n$ typically in the format of

$$M\ddot{q} + Kq = Q \tag{18}$$

that fully describes the dynamical behavior of a mechanical or flexible structural system.

Examples

Example 1: linear three degree of freedom system

Consider the three-mass system depicted in Figure 1. Using Lagrange's method, the equations of motion for this 3 degree of freedom system (3DOF) can be derived as follows.

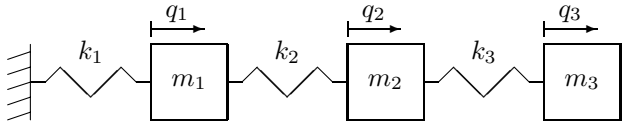

Figure 1: Schematics of linear three degree of freedom system

1. As generalized coordinates for this 3DOF system, simply the three (independent) positions q_i of the masses $m_i,\ i = 1, 2, 3$ can be chosen. With this choice we see that (10) reduces to $r_j = q_j$ and r_j is a scalar.

2. For the kinetic energy T we can directly use (12), as $v_k = \dot{r}_j = \dot{q}_j$ so T is expressed in generalized coordinates

$$T = \frac{1}{2}\sum_{k=1}^{3} m_k \dot{q}_k = \frac{1}{2}m_1\dot{q}_1 + \frac{1}{2}m_2\dot{q}_2 + \frac{1}{2}m_3\dot{q}_3 \tag{19}$$

   and it can be seen that T depends only on $\dot{q}$ and not on q. Comparing the above expression with (14) we see that

$$m_{11} = m_1,\ m_{22} = m_2,\ m_{33} = m_3 \text{ and } m_{ij} = 0 \text{ if } i \neq j$$

   making the generalized mass matrix M a diagonal matrix

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{20}$$

3. For the formulation of the potential energy we have to rely on the potential energy of a linear spring element that is given by $\frac{1}{2}ku^2$, where k is the

stiffness of the spring and u is the deformation of the spring. Using this information we directly see that U satisfies

$$U = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 + \frac{1}{2}k_3(q_3 - q_2)^2 \tag{21}$$

Comparing this expression with (15) it can be see that the entries k_{ij} of the generalized stiffness matrix K are given by

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k + 3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{22}$$

4. Since no external forces are acting on the system, the generalized forces $Q_i = 0$.

With the above information, Lagrange's equations can be formulated. Substitution of (19) and (21) in (7) for $i = 1$ yields

$$\begin{gathered} \frac{\partial T}{\partial \dot{q}_1} = m_1 \dot{q}_1 \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1 \\ \frac{\partial T}{\partial q_1} = 0 \\ \frac{\partial U}{\partial q_1} = (k_1 + k_2)q_1 - k_2 q_2 \end{gathered}$$

creating the first Lagrange equation

$$m_1 \ddot{q}_1 - 0 + (k_1 + k_2)q_1 - k_2 q_2 = 0 \tag{23}$$

Substitution of (19) and (21) in (7) for $i = 2$ yields

$$\begin{gathered} \frac{\partial T}{\partial \dot{q}_2} = m_2 \dot{q}_2 \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2 \ddot{q}_2 \\ \frac{\partial T}{\partial q_2} = 0 \\ \frac{\partial U}{\partial q_2} = (k_2 + k_3)q_2 - k_3 q_3 \end{gathered}$$

creating the second Lagrange equation

$$m_2 \ddot{q}_2 - 0 + (k_2 + k_3)q_2 - k_3 q_3 = 0 \tag{24}$$

Similarly, for $i = 3$ we find

$$\begin{gathered} \frac{\partial T}{\partial \dot{q}_3} = m_3 \dot{q}_3 \;\Rightarrow\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_3 \ddot{q}_3 \\ \frac{\partial T}{\partial q_3} = 0 \\ \frac{\partial U}{\partial q_3} = -k_3 q_2 + k_3 q_3 \end{gathered}$$

creating the third and last Lagrange equation

$$m_3\ddot{q}_3 - 0 - k_3 q_2 + k_3 q_3 = 0 \tag{25}$$

The three Lagrange equations in (23), (24) and (25) can be combined in matrix format:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which indeed resembles the format depicted in (18) with the generalized mass matrix M and stiffness matrix K respectively given in (20) and (22).

Example 2: spring connected mass and inertia with external moment

Consider the system depicted in Figure 2 of a spring k loaded mass m with a spring k connection to an inertia J on which a moment $M(t)$ can be applied.

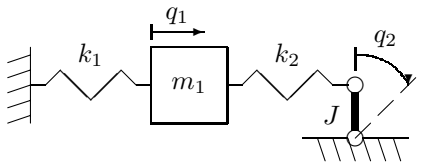

Figure 2: Schematics of spring connected mass and inertia with external moment

1. Obviously, the generalized coordinates are given as the linear position q_1 of the mass m and the rotational position q_2 of the inertia J.

2. The formulation of the kinetic energy is straightforward with $r_j = q_j$, making
$$T = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} J \dot{q}_2^2$$

3. The potential energy is solely due to the presence of the two springs and under the assumption of small rotations q_2 (ignoring higher order effects) we have
$$U = \frac{1}{2} k q_1^2 + \frac{1}{2} k (r q_2 - q_1)^2$$

4. Due to the external moment $M(t)$, the generalized forces $Q_i,\ i = 1, 2$ have to be computed. In equilibrium we see that the total virtual work is given by
$$\delta W = M(t)\delta q_2 \ \Rightarrow\ Q_1 = 0,\ Q_2 = M(t)$$

Combining the above information leads to the following two Lagrange's equations for $i = 1, 2$:

$$\begin{aligned} m\ddot{q}_1 + 2kq_1 - krq_2 &= 0 \\ J\ddot{q}_2 - krq_1 + kr^2 q_2 &= M(t) \end{aligned}$$

which can be combined in the matrix representation

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2k & -kr \\ -kr & kr^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ M(t) \end{bmatrix}$$

Example 3: simple two-story building

Consider the simplified model of a flexible two-story building depicted in Figure 3, whose foundation is spring connected and subjected to a translation force $F(t)$. The model is only a simplified version of the flexibility in a similar building, as only horizontal elastic displacement of the floors is assumed.

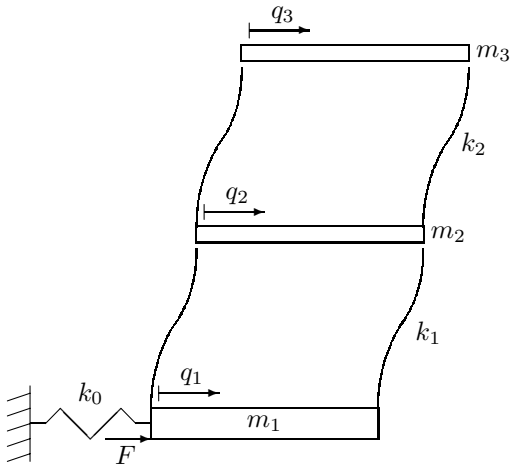

Figure 3: Schematics of simple spring loaded two-story building subjected to base force excitation

1. The generalized coordinates $q_i,\ i = 1, 2, 3$ are chosen as the absolute horizontal position/displacement of the floors.

2. The kinetic energy T is determined by the total velocity $\dot{q}_i$ and the mass of each floor, making

$$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \frac{1}{2} m_3 \dot{q}_3^2 \tag{26}$$

3. Assuming linear shear-stiffness of the elastic side walls, the potential energy U of the structure can be described in the generalized coordinates as follows

$$U = \frac{1}{2} k_0 q_1^2 + \frac{1}{2} k_1 (q_2 - q_1)^2 + \frac{1}{2} k_2 (q_3 - q_2)^2 \tag{27}$$

4. Due to the external force $F(t)$, the generalized forces $Q_i,\ i = 1, 2$ have to be computed. In equilibrium we see that the total virtual work is given by

$$\delta W = F(t)\delta q_1 \ \Rightarrow\ Q_1 = F(t),\ Q_2 = 0,\ Q_3 = 0$$

The above information can be used for formulate the scalar Lagrange's equations for $i = 1, 2, 3$. Substitution of (26) and (27) in (7) for $i = 1$ yields

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_1} = m_1\dot{q}_1 \ \Rightarrow\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) &= m_1\ddot{q}_1 \\ \frac{\partial T}{\partial q_1} &= 0 \\ \frac{\partial U}{\partial q_1} &= (k_0 + k_1)q_1 - k_1 q_2 \end{aligned}$$

creating the first Lagrange's equation

$$m_1\ddot{q}_1 + (k_0 + k_1)q_1 - k_1 q_2 = F(t) \tag{28}$$

For $i = 2$ we have

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_2} = m_2\dot{q}_2 \ \Rightarrow\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) &= m_2\ddot{q}_2 \\ \frac{\partial T}{\partial q_2} &= 0 \\ \frac{\partial U}{\partial q_2} &= -k_1 q_1 + (k_1 + k_2)q_2 - k_2 q_3 \end{aligned}$$

creating the second Lagrange's equation

$$m_2\ddot{q}_2 - k_1 q_1 + (k_1 + k_2)q_2 - k_2 q_3 = 0 \tag{29}$$

And finally for $i = 3$ we have

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_3} = m_3\dot{q}_3 \ \Rightarrow\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) &= m_3\ddot{q}_3 \\ \frac{\partial T}{\partial q_3} &= 0 \\ \frac{\partial U}{\partial q_3} &= -k_2 q_2 + k_2 q_3 \end{aligned}$$

creating the third and last Lagrange's equation

$$m_3\ddot{q}_3 - k_2 q_2 + k_2 q_3 = 0 \tag{30}$$

The three Lagrange equations in (28), (29) and (30) can be combined in a matrix representation

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} F(t) \tag{31}$$

that describes the dynamic behavior of the flexible two-story building.

Mode Shapes and Transfer Functions

Mode Shape Computation

It is clear from the simple examples above that the Lagrange's method leads to a set of *coupled second order differential equations* of the form

$$M\ddot{q}(t) + Kq(t) = Qu(t),\ M = M^T > 0,\ K = K^T \geq 0 \tag{32}$$

where M is a symmetric positive definite generalized mass matrix, K is a symmetric positive generalized stiffness matrix and Q is the generalized input matrix due to an external input force $u(t)$. For this special class of dynamic systems it is possible to also write down a set of *decoupled second order differential equations* by means of a coordinate transformation

$$q(t) := Pp(t) \tag{33}$$

of the generalized coordinates $q(t)$. The existence if a matrix P in the coordinate transformation that is able to decouple the Langrange's equations is based on the Linear Algebra result that for any two real symmetric matrices M and K with $M > 0$, there always exists a non-singular matrix P such that

$$P^T M P = I,\ P^T K P = \Omega^2 = \text{diagonal matrix}$$

Post-multiplication of (32) with P^T and substitution of the coordinate transformation (33) in (32) leads to

$$P^T[MP\ddot{p}(t) + KPp(t) = Qu(t)] \ \Rightarrow\ \ddot{p}(t) + \Omega^2 p(t) = \bar{Q}u(t) \tag{34}$$

which is a set of *decoupled second order differential equations*.

Interesting properties of the dynamic system can be derived from the set of equations in (34) and the coordinate transformation (33):

1. Since Ω^2 is a diagonal matrix, we have a set of decoupled second order differential equations

   $$\ddot{p}_i(t) + \omega_i^2 p_i(t) = \bar{q}_i u(t)$$

   for which the homogeneous solution ($u(t) = 0$) is given by

   $$p_i(t) = \sin(\omega_i t)$$

   As a result, the diagonal elements ω_i of Ω contain the *resonance frequencies* of the mechanical or flexible structural system.

2. A second interesting property that can be derived from (34) are the *eigenmodes* of the system. Since (34) is a set of decoupled equations, an initial condition on the jth element of $p(0)$:

   $$\dot{p}(0) = 0,\quad p(0) = \begin{bmatrix} p_1(0) \\ \vdots \\ p_n(0) \end{bmatrix} \text{ with } p_i(0) = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

will lead to dynamic response $p(t)$ in which *only* the jth element of $p(t)$ is non-zero. As a result, an initial condition on the jth element of $p(0)$ the system remains in the *same direction* when considering the response $p(t)$ as an n-dimensional time dependent vector. Since the direction is maintained we have $p(t) = yp(0)$ at any given time, making

$$q_j = Pp(0) = j\text{th column in P}$$

an eigenmode of the structure. Hence, all eigenmodes are given by the columns of the coordinate transformation P in (33).

For the the diagonal generalized mass matrix M of example 3: simple two-story building it can be see that the matrix

$$\Sigma = \begin{bmatrix} 1/\sqrt{m_1} & 0 & 0 \\ 0 & 1/\sqrt{m_2} & 0 \\ 0 & 0 & 1/\sqrt{m_3} \end{bmatrix}$$

satisfies $\Sigma^T M \Sigma = I$, but the analytic computation of the matrix P that satisfies $P^T M P = I$ and $P^T K P = \Sigma^2$ is more involved. A numerical implementation of the computation can be done with the generalized eigenvalue routine of Matlab. For a given M and K matrix that satisfy $m = M^T > 0$ and $K = K^T \geq 0$, the Matlab command

```
[P,D] = eig(K,M,'chol')
```

will compute the coordinate transformation matrix `P`$= P$ and the matrix `D`$= \Omega^2$ such that $P^T M P = I$ and $P^T K P = \Omega^2$. As an illustrative example we consider the numerical values $m_1 = 10$, $m_2 = 1$, $m_3 = 1$ and $k_0 = 10$, $k1 = 1$, $k_2 = 1$, making

$$M = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 11 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{35}$$

and yielding

$$P \approx \begin{bmatrix} 0.0707 & -0.3035 & -0.0540 \\ 0.5347 & -0.0256 & 0.8446 \\ 0.8149 & 0.2802 & -0.5074 \end{bmatrix}, \; \Omega^2 \approx \begin{bmatrix} 0.3438 & 0 & 0 \\ 0 & 1.0915 & 0 \\ 0 & 0 & 2.6646 \end{bmatrix}$$

As a result, the first resonance mode at $\sqrt{0.3438} \approx= 0.5864$ rad/sec has a shape in the generalized coordinates $q_i,\ i = 1, 2, 3$ given by the first column of P. Similarly, the second resonance mode at $\sqrt{1.0915} \approx= 1.0448$ rad/s has a shape in the generalized coordinates $q_i,\ i = 1, 2, 3$ given by the second column of P. Figure 4 gives an interpretation of the mode shapes for the numerical values in (35) by plotting the three modes shapes and interpolating the intermediate points via a spline interpolation.

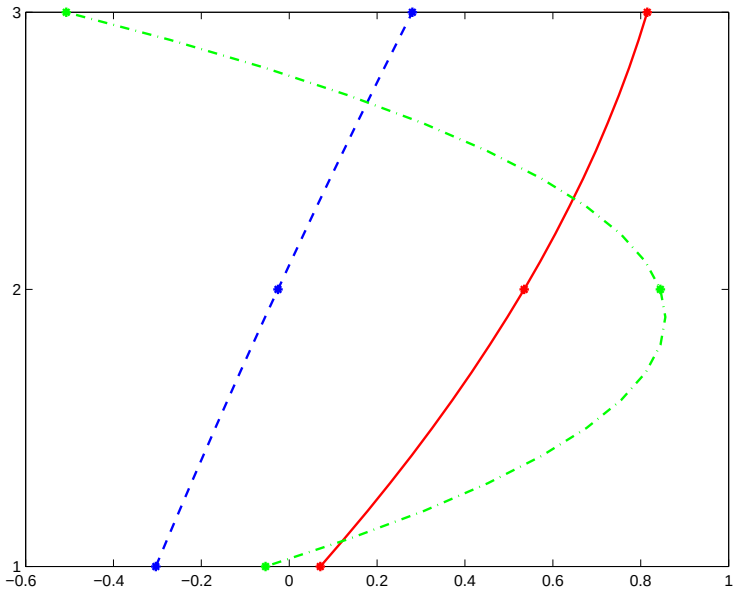

Figure 4: Mode shapes (first = solid, second = dashed, third = dash-dotted) for the numerical values in (35) by spline interpolation

Transfer function representation

Next to the resonance modes and the mode shapes of the structure it is interesting to study the zeros, "anti-resonance modes" or blocking properties of the mechanical or flexible structural system. For relative low order dynamical systems a convenient way to study these properties is to convert the second-order differential equation to a transfer function representation via a Laplace transform. This will allow you to study the dynamic transfer function $H(s)$ between various generalized coordinates $q_i(s),\ i = 1, 2, \ldots, n$.

Application of the Laplace transform to (32) yields

$$[Ms^2 + K]q(s) = Qu(s)$$

which leads to the transfer function

$$G(s) := [Ms^2 + K]^{-1}Q, \quad q(s) = G(s)u(s)$$

that relates the external input forces to the generalized coordinates. When only the jth generalized coordinate is of importance, $G_j(s)$ can be computed via selection of the jth row in $G(s)$:

$$G_j(s) = [0 \ \cdots \ 0 \ 1 \ 0 \cdots \ 0]G(s)$$

and in case $G_i(s)$ and $G_j(s)$ are scalar, the transfer function between to generalized coordinates is defined as

$$q_j(s) = \frac{G_j(s)}{G_i(s)} q_i(s) := H_{ji}(s) q_i(s)$$

With

$$q_j(s) = G_j(s) u(s)$$

any value for $z \in \mathbb{C}$ for which $G_j(z) = 0$ is a zero of $G_j(s)$. In case z satisfies $z \pm j\omega_z$ (for a system without damping), the system has a "anti-resonance mode" at ω_z and will block a sinusoidal input $u(t) = \sin \omega_z t$ at the generalized coordinate $q_j(t)$. Similar argumentation also holds for

$$q_j(s) = H_{ji}(s) q_i(s)$$

when when examining the transfer function $H_{ji}(s)$.

As an example, consider the generalized mass matrix M and generalized stiffness matrix K of example 1: spring connected mass and inertia with external moment. In that case Laplace transformation yields

$$\begin{bmatrix} ms^2 + 2k & -kr \\ -kr & Js^2 + kr^2 \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} M(s)$$

and

$$\begin{aligned} G(s) &= \begin{bmatrix} ms^2 + 2k & -kr \\ -kr & Js^2 + kr^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\det G(s)} \begin{bmatrix} Js^2 + kr^2 & kr \\ kr & ms^2 + 2k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &\quad \text{where } \det G(s) = (Js^2 + kr^2)(ms^2 + 2k) - k^2 r^2 \end{aligned}$$

As a result, the transfer function $G_2(s)$ from the external moment $M(s)$ to the generalized coordinate $q_2(s)$ is given by

$$G_2(s) = \frac{ms^2 + 2k}{Jms^4 + k(2J + r^2 m)s^2 + k^2 r^2}$$

in which a zero or anti-resonance mode can be detected at

$$ms^2 + 2k = 0 \;\Rightarrow\; s = \pm j\omega, \text{ with } \omega = \sqrt{\frac{2k}{m}}$$

Moreover, the transfer function from $q_2(s)$ to $q_1(s)$ is given by

$$q_1(s) = \frac{G_1(s)}{G_2(s)} q_2(s) = H_{12}(s) q_2(s), \; H_{12}(s) = \frac{kr}{ms^2 + 2k}$$

Similar computations can also be done for example 3: simple two-story building, but this would require the analytic computation of the inverse of a 3×3 matrix $[Ms^2 + K]$. In case only specific transfer function such as $H_{12}(s)$ from $q_1(s)$ to $q_2(s)$ needs to be computed, direct computation of $H_{12}(s)$ from the Laplace transformation of (31) is more straightforward.