Main Objectives of Laboratory Experiment:

modeling and feedback control of a gyroscopic system

- rotor drum motor 2 with encoder 2
- encoder 3
- rotor (body D)
- rotor drum (body C)
- brake for axis 3
- rotor motor 1 with encoder 1
- inner gimbal (body B)
- outer gimbal (body A)
- encoder 4 and brake for axis 4
Main Objectives of Laboratory Experiment:

**modeling and feedback control of a gyroscopic system**

Overview of experiments can also be found in a short (online) movie:

download at [http://maecourses.ucsd.edu/labcourse/movies/Gyroscope_Control.WMV](http://maecourses.ucsd.edu/labcourse/movies/Gyroscope_Control.WMV)

Main Objectives of Laboratory Experiment:

**modeling and feedback control of a gyroscopic system**

Ingredients:

- modeling of dynamic behavior of moment exchange gyro
- estimation of model parameters using lumped mass system
- application of control theory for servo/positioning control
- design, implementation & verification of control
- sensitivity and error analysis

Background Theory:

- Kinematics and Newton’s Law \( F = ma \),
- Ordinary Differential Equations (derivation & solutions)
- Linear System Theory (Laplace transform, Transfer function, Bode plots)
- Proportional, Integral and Derivative (PID) control analysis and design (root-locus, Nyquist stability criterion)
Outline of this lecture

- purpose of control & aim of lab experiment
- hardware description
  - schematics
  - hardware in the lab
- theory on modeling
  - modeling a moment exchange gyro
  - step response of a 1st order system
  - step response of a 2nd order system
- outline of laboratory work
  - estimation of parameters: experiments
  - validation of model: simulation & experiments
  - design and implementation of controllers: P- & PD- & PID
- summary
- what should be in your report

Purpose of Control & Some Applications

Application of automatic control: to alter dynamic behavior of a system and/or reduce effect of disturbances.

- industrial processes
  - thickness control of steel plates in a rolling-mill factory
  - consistency control in papermaking machines
  - size and thickness control in glass production processes
  - control of chemical, distillation or batch reactors

- electromechanical systems
  - anti-lock brakes, cruise control and emission control
  - position control in optical or magnetic storage media
  - accurate path execution for robotic systems
  - vibration control in high precision mechanical systems

- aerospace and aeronautical systems
  - flight control of pitch, roll and angle-of-attack in aeroplanes
  - reduction of sound and vibration in helicopters and planes
  - gyroscope for altitude control of satellites
Aim of Lab Experiment

Focus on a (relatively simple) mechanical system of rotating masses/inertias connected in gimbals. Objective is to create a stabilizing feedback system to position mass/inertia at a specified location within a certain time and accuracy.

Control is needed to reduce:

- stabilize rotation (progression of gyro)
- reduce oscillatory behavior (nutation frequency of gyro)

Aim of the experiment:

- insight in control system principles
- design and implement control system
- evaluation of stability & performance
- error analysis and robustness

Schematics of Hardware Description – block diagram

Feedback is essential in control to address stability, disturbance rejection and robustness.

For implementation of feedback: gyroscope G is equipped with sensors (to measure signals) and actuators (to activate system)

For flexibility of control system K: “computer control” or “digital control” and is a combination of:

- ADC (analogue to digital converter)
- DSP (digital signal processor)
- DAC (digital to analogue converter)
**Detailed Hardware Description – components**

- **plant** or **partial gyroscope** (map from input $u$ to output $y$) (actuator, inertias, gimbals and encoder)
- **real-time controller** (ADC, DSP, DAC, amplifier)
- **Personal Computer** (PC) (to run ECP-software and to program DSP)

---

**Hardware in the Lab – moment gyroscope**

Gimbal: pivoted support that allows rotation about a single axis.

Gyroscope: 4 rotating inertias/bodies mounted in gimbals.

- **rotor drum motor 2** with **encoder 2**
- **encoder 3**  
- **rotor (body D)**
- **rotor drum (body C)**
- **brake for axis 3**
- **rotor motor 1 with encoder 1**
- **inner gimbal (body B)**
- **outer gimbal (body A)**
- **encoder 4 and brake for axis 4**
Hardware in the Lab – moment gyroscope

- **Electro-mechanical brakes** to restrict axis rotation

- **Inputs:**
  - $u = \text{voltage } V_1$ to rotor motor 1 OR
  - $u = \text{voltage } V_2$ to rotor drum motor 2

- **Outputs:**
  - $y = \text{angle } \theta_2$ from encoder 2 (rotor drum) OR
  - $y = \text{angle } \theta_3$ from encoder 3 (inner gimbal) OR
  - $y = \text{angle } \theta_4$ of encoder 4 (outer gimbal)

---

Hardware in the Lab – real-time control system

- **real-time controller:** To implement control algorithm and perform digital signal processing. Contains ADC, DSP, DAC & amplifiers.

- **host-PC:** To interact with DSP (run ECP software) and MatLab software.
**Background theory:** modeling a moment exchange gyro

Important parameters (notation) for inertia modeling:

<table>
<thead>
<tr>
<th>body</th>
<th>definition</th>
<th>angular position</th>
<th>inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>outer gimbal</td>
<td>$\theta_4$</td>
<td>$I^A$</td>
</tr>
<tr>
<td>B</td>
<td>inner gimbal</td>
<td>$\theta_3$</td>
<td>$I^B$</td>
</tr>
<tr>
<td>C</td>
<td>rotor drum</td>
<td>$\theta_2$</td>
<td>$I^C$</td>
</tr>
<tr>
<td>D</td>
<td>rotor</td>
<td>$\theta_1$</td>
<td>$I^D$</td>
</tr>
</tbody>
</table>

$$I^b = \begin{bmatrix} I_b & 0 & 0 \\ 0 & J_b & 0 \\ 0 & 0 & K_b \end{bmatrix}, \ b = \text{body } A, B, C \text{ or } D$$

where $I_b$ denotes the inertia along the $x_1$-axis, $J_b$ denotes the inertia along the $x_2$ axis and $K_b$ denotes the inertia along the $x_3$ axis for $b = A, B, C, D$. 
Background theory: modeling a moment exchange gyro

Non-linear equations of motion are of the form:

\[
T_1 = f_1(\theta_2, \theta_3, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) \quad (1)
\]

\[
T_2 = f_2(\theta_2, \theta_3, \omega_1, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2) \quad (2)
\]

\[
0 = f_3(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) \quad (3)
\]

\[
0 = f_4(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4) \quad (4)
\]

and are very hard to work with!

Simplification we will use during experiments in the lab:

Make approximation by considering small perturbations around:

\[
\omega_1 = \Omega \quad \text{(fixed rotational speed of rotor)}
\]

\[
\theta_2 = \bar{\theta}_2 \quad \text{(initial position of rotor drum)}
\]

\[
\theta_3 = \bar{\theta}_3 \quad \text{(initial position of inner gimbal)}
\]

for model approximation by linearization.
**Background theory:** modeling a moment exchange gyro

Example of derivation with effects of linearization:

\[ J_D \dot{\omega}_1 = T_1 - J_D \cos \bar{\theta}_2 \dot{\omega}_3 - J_D \sin \bar{\theta}_2 \cos \bar{\theta}_3 \dot{\omega}_4 \] (1)

\[ (I_C + I_D) \dot{\omega}_2 = T_2 - J_D \Omega \sin \bar{\theta}_2 \omega_3 + J_D \Omega \cos \bar{\theta}_2 \cos \bar{\theta}_3 \dot{\omega}_4 + (I_C + I_D) \sin \bar{\theta}_3 \dot{\omega}_4 \] (2)

\[ (J_B + J_C + J_D - (J_C + J_D - I_D - K_C) \sin^2 \bar{\theta}_2) \dot{\omega}_3 = -J_D \cos \bar{\theta}_2 \dot{\omega}_1 + J_D \Omega \sin \bar{\theta}_2 \omega_2 - J_D \Omega \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dot{\omega}_4 - \sin \bar{\theta}_2 \cos \bar{\theta}_2 \cos \bar{\theta}_3 \] (3)

\[ (I_B + I_C - K_B - K_C - (J_C + J_D - I_D - K_C) \sin^2 \bar{\theta}_2) \sin^2 \bar{\theta}_3 \dot{\omega}_4 = -J_D \sin \bar{\theta}_2 \cos \bar{\theta}_3 \omega_1 - J_D \Omega \cos \bar{\theta}_2 \cos \bar{\theta}_3 \dot{\omega}_2 + J_D \Omega \sin \bar{\theta}_2 \sin \bar{\theta}_3 \dot{\omega}_4 - \sin \bar{\theta}_2 \cos \bar{\theta}_2 \cos \bar{\theta}_3 \omega_3 - (I_D + K_A + K_B + K_C + (J_C + J_D - I_D - K_C) \sin^2 \bar{\theta}_2) \] (4)

Looks complicated, don’t expect you to remember this...

However, note: *all red terms* are constant! Basically:

4 coupled linear equations that relate \( \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4 \) to the torques \( T_1 \) and \( T_2 \).

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**Background theory:** modeling a moment exchange gyro

The 4 coupled linear equations that relate \( \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4 \) to the torques \( T_1 \) and \( T_2 \) can be written in the matrix notation

\[ \mathbf{J} \dot{\omega}(t) = \mathbf{D} \omega(t) + \mathbf{Q} T(t) \]

where the vector \( \omega(t) \) of rotational speeds is given by

\[ \omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \\ \omega_4(t) \end{bmatrix} \]

and \( T(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} \)

and where \( \mathbf{J}, \mathbf{D} \) and \( \mathbf{Q} \) are all previous *red terms* based on a linearization around

\[ \omega_1 = \Omega \] (fixed rotational speed of rotor)
\[ \theta_2 = \bar{\theta}_2 \] (initial position of rotor drum)
\[ \theta_3 = \bar{\theta}_3 \] (initial position of inner gimbal)
**Background theory:** modeling a moment exchange gyro

Choose: $\omega_1 = \Omega$ (constant rotor speed)  
$\theta_2 = \bar{\theta}_2 = 0$ (rotor drum perpendicular)  
$\theta_3 = \bar{\theta}_3 = 0$ (inner gimbal perpendicular)

then the 4 coupled linear differential equations reduce to

\[
J_D \dot{\omega}_1 + J_D \dot{\omega}_3 = T_1 \\
(I_C + I_D) \dot{\omega}_2 = J_D \Omega \omega_4 + T_2 \\
(J_B + J_C + J_D) \dot{\omega}_3 + J_D \dot{\omega}_1 = 0 \\
(I_D + K_A + K_B + K_C) \dot{\omega}_4 = -J_D \Omega \omega_2
\]

and we see $J \dot{\omega}(t) = D \omega(t) + QT(t)$ with

\[
J = \begin{bmatrix}
J_D & 0 & 0 & 0 \\
0 & (I_C + I_D) & 0 & 0 \\
J_D & 0 & (J_B + J_C + J_D) & 0 \\
0 & 0 & 0 & (I_D + K_A + K_B + K_C)
\end{bmatrix}, \\
D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_D \Omega \\
0 & 0 & 0 & 0 \\
0 & -J_D \Omega & 0 & 0
\end{bmatrix}, \\
Q = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

If we extend the vector $\omega(t)$ of angular velocities to also contain (some of) the angular positions

\[
x(t) = \begin{bmatrix}
\theta_2(t) & \theta_3(t) & \theta_4(t) & \omega_1(t) & \omega_2(t) & \omega_3(t) & \omega_4(t)
\end{bmatrix}^T
\]

then we can write down a so-called *state space model*

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t)
\]

where $u(t)$ are inputs (torques by motor 1 or motor 2) and $y(t)$ are outputs (angular rotation from encoders).

In our case matrix $C$ will be

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\] for $\theta_2$, $\theta_3$ and $\theta_4$ as output

With $\dot{\theta}_i(t) = \omega_i(t)$ and $J \dot{\omega}(t) = D \omega(t) + QT(t)$ we see

\[
A = \begin{bmatrix}
0_{3\times3} & U_{3\times4} \\
0_{4\times4} & J^{-1}D
\end{bmatrix}, \\
B = \begin{bmatrix}
0_{3\times3} \\
J^{-1}Q
\end{bmatrix}, \\
U_{3\times4} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
**Background theory:** modeling a moment exchange gyro

Format of **linearized** model of gyro in state space form:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),
\]

where

\[
u(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \end{bmatrix},
\]

\[
x(t) = \begin{bmatrix} \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \\ \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \\ \omega_4(t) \end{bmatrix}^T \text{ and}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -J_\Omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\frac{J_\Omega}{J_c + J_t} & 0 \\
0 & \frac{1}{J_c + J_t} \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Resulting model of gyro is a linear and multivariable model.

**Further assumptions** for your laboratory experiment:

- Voltage applied to **motor 1 & motor 2** can be used to create external torque \( T_1 \) and \( T_2 \).
- Angular positions \( \theta_i = \text{angular position for } i = 2, 3, 4 \) can be measured via **angular encoders**.

**Further simplifications** for your laboratory experiment:

- Lock one or more axis using **brake switches for axis 3 or 4** and **virtual brake for axis 2**.
- Consider only **ONE** motor to create torque \( T_j \) and **ONE** measurement \( \theta_i \) at the time for simplified Single Input Single Output (SISO) control.
**Background theory**: modeling a moment exchange gyro

Impact of this simplification:
- Using only ONE motor to create a torque $T_j$ and ONE measurement $\theta_i$ at the time, reduces the multivariable model to a different SISO models.
- This SISO model can be written as a transfer function $G_{ij}(s)$

**MAIN RESULT**
Let $C_i$ be the $i$th row of the matrix $C$ and let $B_j$ denotes the $j$th column of the matrix $B$, then

$$G_{ij}(s) = C_i(sI - A)^{-1}B_j$$

where $i = 2, 3, 4$ due to encoder 2, 3 or 4 and $i = 1, 2$ due to motor input 1 or 2.

Example: transfer function from motor 2 to encoder 4

$$\theta_4(s) = G_{42}(s)T_2(s), \quad G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} \frac{1}{s}$$

---

**Background theory**: modeling a moment exchange gyro

- **motor 2** & **encoder 2**
- **encoder 3**
- **rotor (body D)**
- **rotor drum (body C)**
- **brake for axis 3**
- **motor 1** & **encoder 1**
- **inner gimbal (body B)**
- **outer gimbal (body A)**
- **encoder 4** with **brake for axis 4**

Configurations of the simplified gyroscope during your lab:

- **Week 1**: brake 4 on and (virtual) brake 2 on.
  Measure $\theta_3$, control motor 1: inertial control.
- **Week 2**: brake 3 on and $\omega_1 = \Omega$.
  Measure $\theta_2$, control motor 2: nutation frequency control.
- **Week 3**: brake 3 on and $\omega_1 = \Omega$.
  Measure $\theta_4$, control motor 2: precession control.
Background theory: modeling a moment exchange gyro

Models during your lab:

- **Week 1**: brake 4 on and (virtual) brake 2 on leads to
  \[ y_3(s) = G_{31}(s)u_1(s), \text{ with } G_{31}(s) = \frac{-1}{(J_B + J_C)s^2} \]

- **Week 2**: brake 3 on and \( \omega_1 = \Omega \): \( y_2(s) = G_{22}(s)u_2(s) \) with
  \[ G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} \]

- **Week 3**: brake 3 on and \( \omega_1 = \Omega \): \( y_4(s) = G_{42}(s)u_2(s) \) with
  \[ G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} \]

**Background theory**: reduction to 1DOF system

Transfer functions \( G_{31}(s), G_{22}(s) \) and \( G_{42}(s) \) look complicated, but are basically of the form:

\[
\begin{align*}
G_{31}(s) &= \frac{-1}{(J_B + J_C)s^2} = \frac{K_0}{s^2 + \beta_0 s} \\
G_{22}(s) &= \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} = \frac{K_1 \omega_n^2}{s^2 + 2 \beta \omega_n s + \omega_n^2} \\
G_{42}(s) &= \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} = \frac{K_2 \omega_n^2}{s^2 + 2 \beta \omega_n s + \omega_n^2} \cdot \frac{1}{s}
\end{align*}
\]

where we have the (lumped) parameters:

- scaling \( K_0 \), (additional) damping \( \beta_0 \)
- scaling \( K_1 \), damping \( \beta \) and nutation frequency \( \omega_n \)
- scaling \( K_2 \), damping \( \beta \) and nutation frequency \( \omega_n \)
**Background theory**: reduction to 1DOF system

To be able to do control design, you need to find the parameters of the transfer functions

\[
G_{31}(s) = \frac{-1}{(J_B + J_C)s^2} = \frac{K_0}{s^2 + \beta_0 s}
\]

\[
G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} = \frac{K_1 \omega_n^2}{s^2 + 2\beta \omega_n s + \omega_n^2}
\]

\[
G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} = \frac{K_2 \omega_n^2}{s^2 + 2\beta \omega_n s + \omega_n^2} \cdot \frac{1}{s}
\]

Hence, for modeling we would need to determine inertia of bodies in three different axes...

Alternative to determining inertia of bodies in three different axes: determine (lumped) parameters \(K_0, K_1, K_2, \beta_0, \beta_1\) and \(\omega_n\) via **dynamic experiments** (step response experiments).

---

**Background theory**: step response of a 1st order system

Let us consider

\[
G_{31}(s) = \frac{K_0}{s^2 + \beta_0 s}
\]

as our model for our 1st week laboratory experiment:

- **Instead** of using the position \(\theta_3(t)\) as output, consider using the velocity \(\omega_3(t)\) as output.
- With \(\omega_3(t) = \frac{d}{dt} \theta_3(t)\) we have \(\omega_3(s) = s \theta_3(s)\) using Laplace.
- **Hence**, choosing \(\omega_3(t)\) as an output \(y_3(t)\) modifies \(G_{31}(s) \cdot s\) to a simple 1st order system:

\[
y_3(s) = \frac{K_0}{s + \beta_0} u_1(s)
\]

You can determine \((K_0, \beta_0)\) from a dynamic (step response) experiment...
Background theory: step response of a 1st order system

**MAIN RESULT**
Consider a 1st order system with a transfer function
\[ G_{ij}(s) = \frac{K}{s + \beta} \]
then a step input \( u(t) = U, \ t \geq 0 \) of size \( U \) results in the output response
\[ y(t) = \frac{K}{\beta} \cdot U \cdot \left[ 1 - e^{-\beta t} \right] \]
Result follows directly from inverse Laplace transform of
\[ y(s) = G_{ij}(s) \cdot \frac{U}{s} = \frac{KU}{s + \beta} \cdot \frac{1}{s} \]

How about the inverse problem of finding \( \beta \) and \( K \) from (velocity) measurement \( y(t) \)?
Background theory: step response of a 1st order system

With \( \left. \frac{d}{dt}y(t) \right|_{t=0} \) and \( y_\infty \) from velocity step response:

\[
\left. \frac{d}{dt}y(t) \right|_{t=0} = \left. K \cdot U e^{-\beta t} \right|_{t=0} = K \cdot U
\]

and hence

\[
\hat{K} = \frac{1}{U} \cdot \left. \frac{d}{dt}y(t) \right|_{t=0}
\]

Knowing that the (velocity) \( \lim_{t \to \infty} y(t) = y_\infty = \frac{K}{\beta} \cdot U \) we find

\[
\hat{\beta} = \frac{\hat{K}}{y_\infty} \cdot U
\]

---

Background theory: step response of a 2nd order system

Let us consider

\[
G_{22}(s) = \frac{K \omega_n^2}{s^2 + 2\beta \omega_n s + \omega_n^2}
\]

as our model for our 2nd week laboratory experiment:

- This is a standard second order system, where \( \omega_n = \) undamped (nutation) frequency, \( 0 \leq \beta \leq 1 \) is the damping ratio, \( K \) is the gain.
- This is similar to the transfer function of a 1DOF mechanical system (with possibly zero damping \( d \))!

\[
m \dddot{x}(t) + d \dot{x}(t) + kx(t) = F(t) \Rightarrow x(s) = \frac{1}{ms^2 + ds + k} F(s)
\]
**Background theory**: step response of a 2nd order system

In case \( k > 0 \) we can write

\[
G(s) = \frac{1}{ms^2 + ds + k} = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}
\]

where

\[
K := \frac{1}{k} \text{ (compliance)}, \\
\omega_n := \sqrt{\frac{k}{m}} \text{ (resonance)}, \\
\beta := \frac{d}{2\sqrt{mk}} \text{ (damping ratio)}
\]

Note the resemblance with the 2nd week lab experiment model

\[
G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega_D^2 J_D^2} = \frac{K_1\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}
\]

You determine \((K_1, \omega_n, \beta)\) from a dynamic (step response) experiment.

---

**Background theory**: step response of a 2nd order system

**MAIN RESULT**

Consider a 2nd order system with a transfer function

\[
G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}
\]

then a step input \( u(t) = U, \ t \geq 0 \) of size \( U \) results in the output response

\[
y(t) = K \cdot U \cdot \left[ 1 - e^{-\beta\omega_n t} \sin(\omega_d t + \phi) \right]
\]

where

\[
\omega_d = \omega_n\sqrt{1 - \beta^2} \quad \text{damped resonance frequency in rad/s} \\
\phi = \tan^{-1}\frac{1 - \beta^2}{\beta} \quad \text{phase shift of response in rad}
\]
**Background theory**: step response of a 2nd order system

**DERIVATION** for 1DOF (second order) system

Compute the dynamic response via inverse Laplace transform!

Consider a step input $u(t) = U, \ t \geq 0$ of size $U$. Then $u(s) = \frac{U}{s}$ and for the 1DOF system we have

$$y(s) = G(s)u(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{U}{s}$$

and the inverse Laplace transform is given by

$$y(t) = K \cdot U \cdot \left[ 1 - e^{-\beta\omega_n t} \sin(\omega_d t + \phi) \right]$$

where

$$\omega_d = \omega_n \sqrt{1 - \beta^2} \quad \text{damped resonance frequency in rad/s}$$

$$\phi = \tan^{-1} \frac{\sqrt{1 - \beta^2}}{\beta} \quad \text{phase shift of response in rad}$$

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**Background theory**: step response of a 2nd order system

Typical picture of $y(t) = K \cdot U \left[ 1 - e^{-\beta\omega_n t} \sin(\omega_d t + \phi) \right]$ for a step size of $U = 1$, scaling $K = 5$, undamped resonance frequency $\omega_n = 2 \cdot 2\pi \approx 12.566$ rad/s and damping ratio $\beta = 0.2$:

How about the reverse problem of finding $\omega_n, \beta$ and $K$ from $y(t)$?

MAE175a Control Experiment, Winter 2014 – R.A. de Callafon – Slide 36
**Background theory**: step response of a 2nd order system

With the times $t_0$, $t_n$ and the values $y_0$, $y_n$ and $y_\infty$ from (position or velocity) step response:

\[
\omega_d = 2\pi \frac{n}{t_n - t_0} \quad \text{(damped resonance frequency)}
\]

\[
\beta \omega_n = \frac{1}{t_n - t_0} \ln \left( \frac{y_0 - y_\infty}{y_n - y_\infty} \right) \quad \text{(exponential decay term)}
\]

where $n =$ number of oscillations between $t_n$ and $t_0$.

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**Background theory**: step response of a 2nd order system

With the estimates

\[
\omega_d = 2\pi \frac{n}{t_n - t_0} \quad \text{(damped resonance frequency)}
\]

\[
\beta \omega_n = \frac{1}{t_n - t_0} \ln \left( \frac{y_0 - y_\infty}{y_n - y_\infty} \right) \quad \text{(exponential decay term)}
\]

we can now compute $\omega_n$ and $\beta$:

\[
\omega_n = \sqrt{\omega_d^2 + (\beta \omega_n)^2} \quad \text{(undamped resonance frequency)}
\]

\[
\beta = \frac{\beta \omega_n}{\omega_n} \quad \text{(damping ratio)}
\]

and $K$ follows from

\[
K = \frac{y_\infty}{U} \quad \text{(compliance)}
\]

MAE175a Control Experiment, Winter 2014 – R.A. de Callafon – Slide 38
Background theory: step response of a 2nd order system

Let us now consider

\[ G_{42}(s) = \frac{K_2\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \]

as our model for our 3nd week laboratory experiment:

- **Instead** of using the position \( \theta_4(t) \) as output, consider using the velocity \( \omega_4(t) \) as output.
- With \( \omega_4(t) = \frac{d}{dt}\theta_4(t) \) we have \( \omega_4(s) = s\theta_4(s) \) using Laplace.
- **Hence**, choosing \( \omega_4(t) \) as an output modifies \( G_{42}(s) \cdot s \) back into a standard 2nd order system

\[ y_4(s) = \frac{K_2\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} u_2(s) \]

**NOTE:** we can use the same step response results and parameter estimation as we did in our 2nd week laboratory experiment to estimate \( K_2, \beta \) and \( \omega_n \).

Outline of Lab Work: estimation of model parameters

Important observation: gyroscope in 1st week lab experiment appears (dynamically) equivalent to a second order system with

\[ G_{ij}(s) = \frac{K_0}{s^2 + \beta_0 s} \]

and can be reduced to a 1st order system

\[ G_{ij}(s) \cdot s = \frac{K_0}{s + \beta_0} \]

when using the velocity \( \omega_3(t) \) as output \( y_3(t) \).

Main Idea:

- Experiment: use a step response laboratory experiment to formulate parameter estimates \( \hat{K} \) and \( \hat{\beta}_0 \).
- Validate model, so that measured step response looks similar to simulated step response
Outline of Lab Work: estimation of model parameters

Important observation: gyroscope in 2nd and 3rd week lab experiment appears (dynamically) equivalent to a single 1DOF with a transfer function

\[ G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \]

where
\[ K := \text{scaling}, \]
\[ \omega_n := \text{nutation frequency}, \]
\[ \beta := \text{damping ratio} \]

Main Idea:

- Experiment: use a step response laboratory experiment to formulate parameter estimates \( \hat{K}, \hat{\omega}_n \) and \( \hat{\beta} \).
- Validate model, so that measured step response looks similar to simulated step response

EXAMPLE: step of \( U = 0.5V \) on motor

Read from plot:
\[ t_0 = 0.25, \ t_n = 1.25, \ y_0 = 7.5, \ y_n = 5.25, \ y_\infty = 5 \]

\[ \hat{\omega}_d = 2\pi \frac{n}{t_n - t_0} = 2\pi \frac{2}{1.25 - 0.25} = 4\pi \]
\[ \hat{\beta}\omega_n = \frac{1}{t_n - t_0} \ln \left( \frac{y_0 - y_\infty}{y_n - y_\infty} \right) = \frac{1}{1.25 - 0.25} \ln \left( \frac{7.5 - 5}{5.25 - 5} \right) = \ln(10) \]
\[ \hat{\omega}_n = \sqrt{\hat{\omega}_d^2 + (\hat{\beta}\omega_n)^2} = \sqrt{16\pi^2 + \ln(10)^2} \approx 12.78 \]
\[ \beta = \frac{\hat{\beta}\omega_n}{\hat{\omega}_n} \approx \frac{\ln(10)}{12.78} \approx 0.18 \]
\[ \hat{K} = \frac{y_\infty}{U} = \frac{5}{0.5} = 10 \]

Units of \( \hat{K} \)? Does it matter?
Outline of Lab Work: estimation of model parameters

Conclusions and summary of work:

• Verify transfer function you are working with (week 1, 2 or 3)
• It will be of the form
  \[ G_{ij}(s) = \frac{K}{s + \beta} \left( \frac{1}{s} \right) \]  or \[ G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_ns + \omega_n^2} \left( \frac{1}{s} \right) \]
• Estimate the parameters model parameters \( (K, \omega_n, \beta) \) via the measurement of a step response.
• Verify the simulation of your step response with the measurement of a step response.
• Substitute parameter values to create your dynamic model \( G(s) \) to be used for control design

Luckily, we have Matlab and a matlab script/function called maelab.m to help you with his.

Simply enter your models in a file called models.m

NOTES:

• During week 1 you DO NOT use a constant rotor speed \( \Omega \), as you will be controlling the rotor motor 1 yourself.
• During week 2 & 3 you DO use a constant rotor speed \( \Omega \).
• For each laboratory section a different constant rotational rotor speed \( \Omega \) is used.
• The rotational rotor speed you will be working with is given in the config.txt or info.txt file.
• Model parameters \( (K, \omega_n, \beta) \) will depend on physical quantities (mass, inertia, damping, etc.) and need to be determined during laboratory.
• Important in validation of laboratory work: comparison of actual (dynamic) experiments with a (dynamic) simulation of your model \( G(s) \) before you start doing control design!
• With a bad model you cannot do a proper control design...
Outline of Lab Work: model validation

Keep in mind:

- From parameter estimation experiments we know obtain a model specified as a transfer function model \( y(s) = G_{ij}(s)u(s) \)

\[
G_{ij}(s) = \frac{K}{s^2 + \beta s} \quad \text{or} \quad G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}\left(\frac{1}{s}\right)
\]

where \( u(s) = \) is control input (motor Voltage or Force) and \( y(t) = \) system output (position in encoder counts)

- Model \( G(s) \) is created automatically for you via `maelab.m` script file by modifying the `models.m` file.

- You are going to use the model \( G(s) \) to design a controller \( K(s) \), so model \( G(s) \) should be validated first!

Outline of Lab Work: model validation

What is suitable for model validation:

- Validate the estimation of each set of model parameters by comparing actual experiments with a simulation of a 1DOF model \( G_{ij}(s) \).

  This can be done with the `maelab.m` script file.

- With a bad (unvalidated) model \( G_{ij}(s) \) you cannot do a proper model-based control \( K(s) \) design!
Outline of Lab Work: design of controllers

Each week you construct feedback loop around $G(s)$:

![Schematic view of closed loop configuration](image)

Find a feedback controller $K(s)$ that satisfies:

- move a mass/inertia to a certain (angular) position as fast as possible
- limit overshoot during control/positioning to 25%
- no steady-state error $e$
- illustrate disturbance rejection when control is implemented

Trade off in design specifications:

- high speed ↔ overshoot
- overshoot ↔ robustness

Outline of Lab Work: design of controllers

Controller configurations that can be implemented:

- P-control
  \[ u(t) = k_p e(t), \quad e(t) = r(t) - y(t) \]

- PD-control
  \[ u(t) = k_p e(t) + k_d \frac{d}{dt} e(t), \quad e(t) = r(t) - y(t) \]

- PID control
  \[ u(t) = k_p e(t) + k_i \int_{\tau=0}^{t} e(\tau) d\tau + k_d \frac{d}{dt} e(t), \quad e(t) = r(t) - y(t) \]
Outline of Lab Work: design of controllers

All controllers are implemented in a discrete-time!

Let $T_s = \text{sample time}$, $t = kT_s$ and $e(k) = r(k) - y(k)$ we get

- P-control
  \[ u(k) = k_p e(k) \]

- PD-control
  \[ u(k) = k_p e(k) + \frac{k_d}{T_s} [e(k) - e(k - 1)] \]

- PID control
  \[ u(k) = k_p e(k) + k_i T_s \sum_{l=0}^{k} e(l) + \frac{k_d}{T_s} [e(k) - e(k - 1)] \]

Templates will be available for you to enter your control parameters $k_p$, $k_d$ and/or $k_i$.

Example of template

```
;***************define user variables ***************
#define kp q2 ; proportional gain
#define kd q3 ; derivative gain
#define ki q4 ; integral gain
#define Ts q5 ; sampling time for scaling of derivative and integral gain
#define kdd q6 ; scaled derivative gain
#define kii q7 ; scaled integral gain
#define enc2_prev q8 ; keep track of previous encoder 2 measurement for derivative
#define error q9 ; keep track of error signal for integration

;***************Initialize variables***************
Ts=.00442 ; For local use only. Must ALSO be set in SETUP CONTROL ALGORITHM dialog box!
kp=0.2 ; set proportional gain
kd=0.02 ; set derivative gain
ki=0.2 ; set integral gain
kdd=kd/Ts ; divide by Ts here to save real-time processing
kii=ki*Ts ; multiply by Ts here to save real-time processing
error=0 ; initialize integrator error

;PID Controller that uses encoder 2 and motor 2 (nutation control)
;*************** real time code which is run every servo period ***
begin
  error=error+cmd2_pos-enc2_pos
  control_effort2=kp*(cmd2_pos-enc2_pos)+kii*error+kdd*(enc2_prev-enc2_pos)
  enc2_prev=enc2_pos
end
```
Outline of Lab Work: design of controllers (loop gain)

Model \( G(s) \) of plant should be used for design of controller \( K(s) \):

\[
\begin{array}{c}
  r \\
  \downarrow \\
  e \\
  \downarrow \\
  K(s) \\
  \downarrow \\
  u \\
  \downarrow \\
  G(s) \\
  \downarrow \\
  y
\end{array}
\]

For design of controller \( K(s) \), consider the loop gain:

\[
L(s) := K(s)G(s)
\]

loop gain: series connection of \( K(s) \) and \( G(s) \)

Dynamics of loop gain \( L(s) \) consists of fixed part \( G(s) \) (plant dynamics) and to-be-designed part \( K(s) \) (controller)

Outline of Lab Work: design of controllers (stability)

Loop gain \( L(s) := K(s)G(s) \) important for:

- Stability
- Design specification

STABILITY:
With closed-loop poles found by those values of \( s \in \mathbb{C} \) that satisfy

\[
1 + L(s) = 0
\]

all closed-loop poles should have negative real values (lie in the left part of the complex plane)

Stability can be checked by:

- Actually computing the solutions to \( L(s) = -1 \) as a function of \( k_p, k_i, k_d \): Root Locus Method
- See if Nyquist plot of \( L(s) \) encircles the point \( -1 \) as a function of \( k_p, k_i, k_d \): Nyquist or Frequency Domain Method
**Outline of Lab Work:** design of controllers (design specs)

![Block diagram](image)

**Error rejection transfer function:**

\[ E(s) = \frac{1}{1 + L(s)} \] (map from \( r \) to \( e \))

To avoid a steady state error \( e(t) \) as \( t \to \infty \), one specification for the loop gain can be found via the **final value theorem**. With \( L(s) = G(s)K(s) \) we have:

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} s \cdot \frac{1}{1 + L(s)} r(s) \]

With \( r(t) = \) (unit) step input, \( r(s) = \frac{1}{s} \) and

\[ \lim_{s \to 0} |L(s)| = \infty \]

is needed for zero steady-state behavior!

---

**Outline of Lab Work:** design of controllers (design specs)

![Block diagram](image)

**Closed loop transfer function:**

\[ T(s) = \frac{L(s)}{1 + L(s)} \] (map from \( r \) to \( y \))

To make sure \( y \) follows \( r \), we would like to make \( T(s) = 1 \) as close as possible.

Notice that with **Error rejection transfer function**:

\[ E(s) = \frac{1}{1 + L(s)} \] (map from \( r \) to \( e \))

we have

\[ T(s) + E(s) = 1 \]

Hence, if you can make \( |E(s)| \approx 0 \) small, then \( |T(s)| \approx 1 \).
**Outline of Lab Work:** design of controllers (graphical design)

Computation of $K(s)$: translate design specifications to $L(s)$, $E(S)$ or $T(s)$ specifications.

- Use graphical analysis and design utilities (root locus or frequency domain methods) to shape loop gain $L(s)$ and design controller $K(s)$.
- Root-locus and frequency domain design method is implemented in Matlab via the `rltool` command.
- Frequency domain design method has also been implemented in a Matlab script file `maelab` provided during the lab.
- Stability via Nyquist criterion (do not encircle point -1)
- Phase and amplitude margin (stability and robustness) translate to shape Bode plot of loop gain $L(s) = G(s)K(s)$:
  - **phase margin:** when $|L(s)| = 1$, $\angle L(s) > -\pi$ rad
  - **amplitude margin:** when $\angle L(s) = -\pi$ rad, $|L(s)| < 1$

---

**Example of figures produced by `maelab` script file**

![Graphical representations](image-url)

- **Nyquist contour** $L(s)$
- **Magnitude** $L(s)$
- **Phase** $L(s)$
- **Closed loop stable**
Effects of control parameters: for PD-control and a standard 2nd order plant model this can be analyzed as follows:

\[ T(s) = \frac{L(s)}{1 + L(s)} = \frac{(k_p + k_ds)s^2 + \omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \]

which yields the closed-loop transfer function

\[ T(s) = \frac{\omega_n^2(k_p + k_ds)}{s^2 + 2\beta\omega_n + \omega_n^2} \]

with \( \bar{\omega}_n = \omega_n\sqrt{1 + k_p} \) and \( \bar{\beta} = \frac{\beta + \omega_n k_d/2}{\sqrt{1 + k_p}} \)

In this case \( T(s) \) is also a second order system and with knowledge of the step response, we can conclude that the following influence of the controller parameters:

- \( k_p \) ↔ speed of response
- \( k_p \) ↔ damping
- \( k_p \) ↔ steady-state error
- \( k_d \) ↔ damping

Outline of Lab Work: design of controllers (summary)

- Model \( G(s) \) of plant should be used for design of your controller \( K(s) \)!
- Think about the controller needed (P, PD or PID) to stabilize your system
- Obviously, controllers \( K(s) \) for week 1, 2 and 3 are all different and \( G(s) \) is different!
- Keep in mind the requirement of 25% overshoot, and no steady state error, e.g. \( r(t) = y(t) \) as \( t \to \infty \).
- Use graphical design tools to design your P, PD and PID control:
  - Root-locus and frequency domain design method is implemented in Matlab via the \texttt{rltool} command.
  - Frequency domain design method also implemented in a Matlab script file \texttt{maelab} provided during the lab.
Outline of Lab Work: design of controllers (summary)

- Consider how the frequency resp of P- and PD- and PID controller modifies the loop gain $L(s) = G(s)K(s)$. Look at asymptotes of Bode plot of controller

$$K(s) = \frac{k_ds^2 + kp + ki}{s}$$

- Phase and amplitude margin (stability and robustness) translate to shape Bode plot of loop gain $L(s) = G(s)K(s)$:
  - phase margin: when $|L(s)| = 1, \angle L(s) > -\pi$ rad
  - amplitude margin: when $\angle L(s) = -\pi$ rad, $|L(s)| < 1$

- Argument and motive your control design in rapport.
- No trial-and-error control design results are accepted.

---

Summary of Lab Work

- **first week**
  Study laboratory handout. Get familiar with the multi-input-multi-output (MIMO) moment gyroscope system, introduction to ECP software used for experiments and controller implementation. Engage axis 4 brake (lock outer gimbal) and virtual brake 2 (lock rotor drum). Design and implement feedback using motor 1 as input and encoder 3 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Design controller based on model in Matlab and implement/verify on actual system.

- **second week**
  Engage axis 3 brake (lock inner gimbal). Bring rotor at fixed speed $\Omega$. Design and implement feedback using motor 2 as input and encoder 2 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Validation of model parameters via comparison of simulation and experiments. Design controller based on model in Matlab and implement/verify on actual system.

- **third week**
  Engage axis 3 brake (lock inner gimbal). Bring rotor at fixed speed $\Omega$. Design and implement feedback using motor 2 as input and encoder 4 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Validation of model parameters via comparison of simulation and experiments. Validation of nutation & precession of MIMO moment gyroscope dynamics. Design controller based on model in Matlab and implement/verify on actual system.
What should be in your report (1-2)

- **Abstract**
  Standalone - make sure it contains clear statements w.r.t motivation, purpose of experiment, main findings (numerical) and conclusions.

- **Introduction**
  - Motivation (why are you doing this experiment)
  - Short description of the main engineering discipline (controls)
  - Answer the question: what is the aim of this experiment/report?

- **Theory**
  - Feedback system
  - Modeling
  - Parameter estimation
  - Control design

What should be in your report (2-2)

- **Experimental Procedure**
  - Short description of experiment
  - How are experiments done (detailed enough so someone else could repeat them)

- **Results**
  - Estimation of parameters $K$, $\omega_n$ and $\beta$
  - Validation of models $G_{ij}(s)$
  - Design of controller $K(s)$ and Implementation

- **Discussion**
  - Why are simulation results different from experiments?
  - Could the model be validated?
  - Are designed controller parameters O.K. from model?

- **Conclusions**

- **Error Analysis**
  - Mean, standard deviation and 99% confidence intervals of estimated parameters $\omega_n$, $\beta$ and $K$ from data