Moment Gyroscope Control Experiment (MAE175a)

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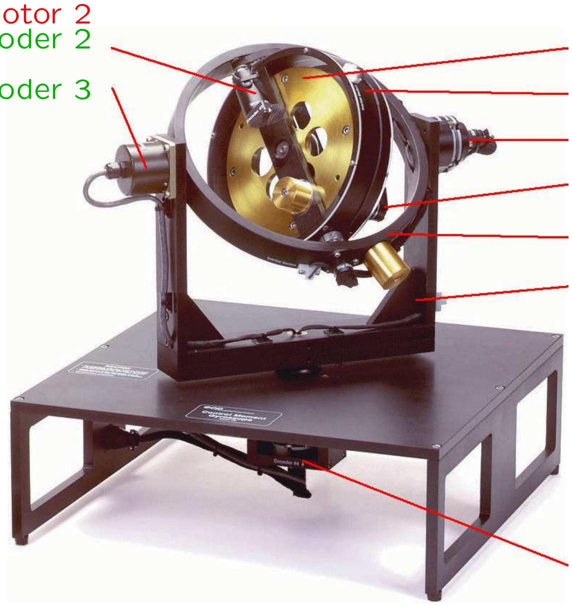
this lecture and lab handouts will be available on
http://maecourses.ucsd.edu/labcourse/

Main Objectives of Laboratory Experiment:

modeling and feedback control of a gyroscopic system



Main Objectives of Laboratory Experiment:

modeling and feedback control of a gyroscopic system

Overview of experiments can also be found in a short (online) movie:

download at http://maecourses.ucsd.edu/labcourse/movies/Gyroscope_Control.WMV

Main Objectives of Laboratory Experiment:

modeling and feedback control of a gyroscopic system

Ingredients:

- modeling of dynamic behavior of moment exchange gyro
- estimation of model parameters using lumped mass system
- application of control theory for servo/positioning control
- design, implementation & verification of control
- sensitivity and error analysis

Background Theory:

- Kinematics and Newton's Law ($F = ma$),
- Ordinary Differential Equations (derivation & solutions)
- Linear System Theory (Laplace transform, Transfer function, Bode plots)
- Proportional, Integral and Derivative (PID) control analysis and design (root-locus, Nyquist stability criterion)

Outline of this lecture

- purpose of control & aim of lab experiment
- hardware description
  - schematics
  - hardware in the lab
- theory on modeling
  - modeling a moment exchange gyro
  - step response of a 1st order system
  - step response of a 2nd order system
- outline of laboratory work
  - estimation of parameters: experiments
  - validation of model: simulation & experiments
  - design and implementation of controllers: P- & PD- & PID
- summary
- what should be in your report

Purpose of Control & Some Applications

Application of automatic control: to alter dynamic behavior of a system and/or reduce effect of disturbances.

- industrial processes
  - thickness control of steel plates in a rolling-mill factory
  - consistency control in papermaking machines
  - size and thickness control in glass production processes
  - control of chemical, distillation or batch reactors
- electromechanical systems
  - anti-lock brakes, cruise control and emission control
  - position control in optical or magnetic storage media
  - accurate path execution for robotic systems
  - vibration control in high precision mechanical systems
- aerospace and aeronautical systems
  - flight control of pitch, roll and angle-of-attack in aeroplanes
  - reduction of sound and vibration in helicopters and planes
  - gyroscope for altitude control of satellites

Aim of Lab Experiment

Focus on a (relatively simple) mechanical system of rotating masses/inertias connected in gimbals. Objective is to *create a stabilizing feedback system* to *position mass/inertia* at a *specified location* within a *certain time* and *accuracy*.

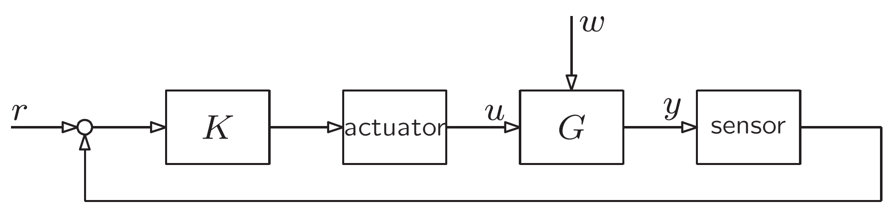
Control is needed to reduce:

- stabilize rotation (progression of gyro)
- reduce oscillatory behavior (nutation frequency of gyro)

Aim of the experiment:

- insight in control system principles
- design and implement control system
- evaluation of stability & performance
- error analysis and robustness

Schematics of Hardware Description – block diagram



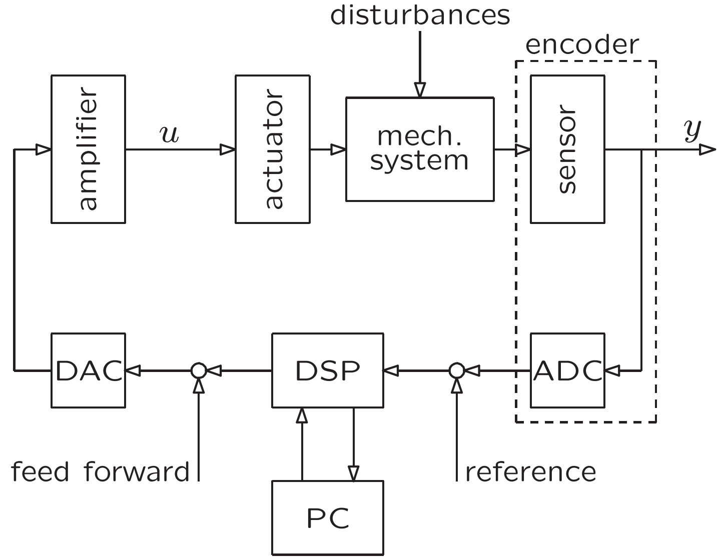
Feedback is essential in control to address *stability, disturbance rejection* and *robustness*.

For implementation of feedback: gyroscope G is equipped with *sensors* (to measure signals) and *actuators* (to activate system)

For flexibility of control system K: "computer control" or "digital control" and is a combination of:

- ADC (analogue to digital converter)
- DSP (digital signal processor)
- DAC (digital to analogue converter)

Detailed Hardware Description – components

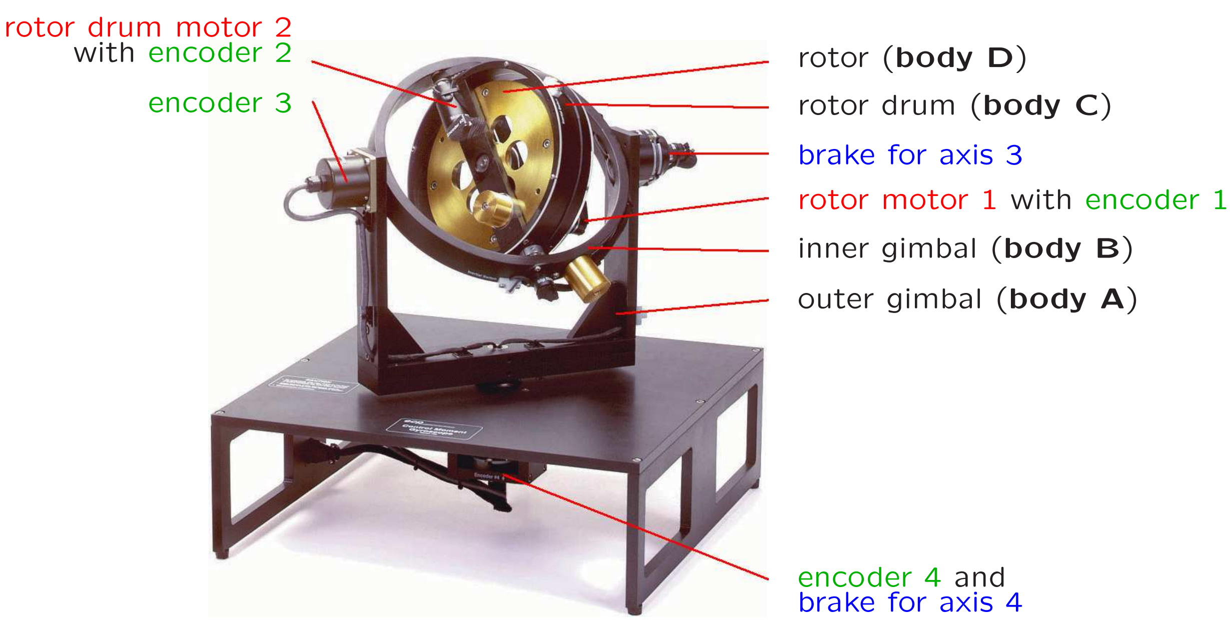


- **plant** or **partial gyroscope** (map from input u to output y)
  (actuator, inertias, gimbals and encoder)
- **real-time controller**
  (ADC, DSP, DAC, amplifier)
- **Personal Computer** (PC)
  (to run ECP-software and to program DSP)

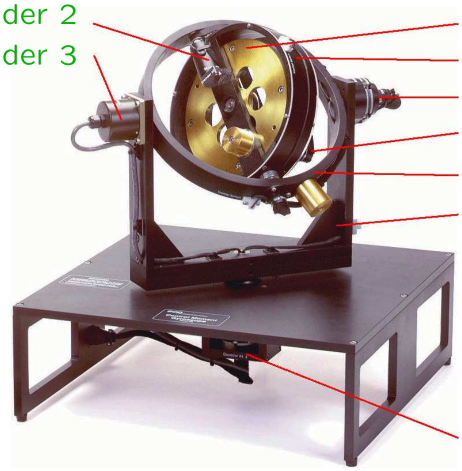
Hardware in the Lab – moment gyroscope

Gimbal: pivoted support that allows rotation about a single axis.

Gyroscope: 4 rotating inertias/bodies mounted in gimbals.

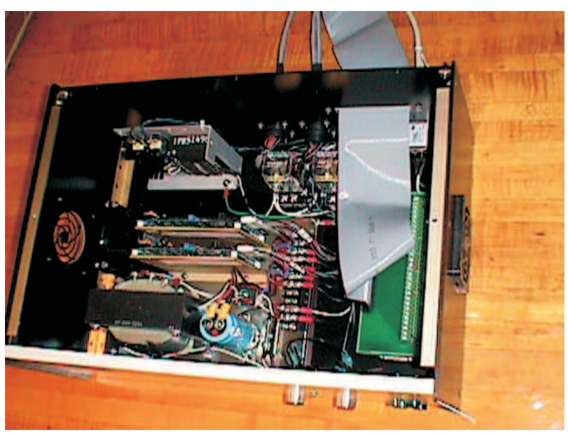


Hardware in the Lab – moment gyroscope



- **Electro-mechanical brakes** to restrict axis rotation
- **Inputs**: $u =$ voltage V_1 to rotor motor 1 OR
  $u =$ voltage V_2 to rotor drum motor 2
- **Outputs**: $y =$ angle θ_2 from encoder 2 (rotor drum) OR
  $y =$ angle θ_3 from encoder 3 (inner gimbal) OR
  $y =$ angle θ_4 of encoder 4 (outer gimbal)

Hardware in the Lab – real-time control system

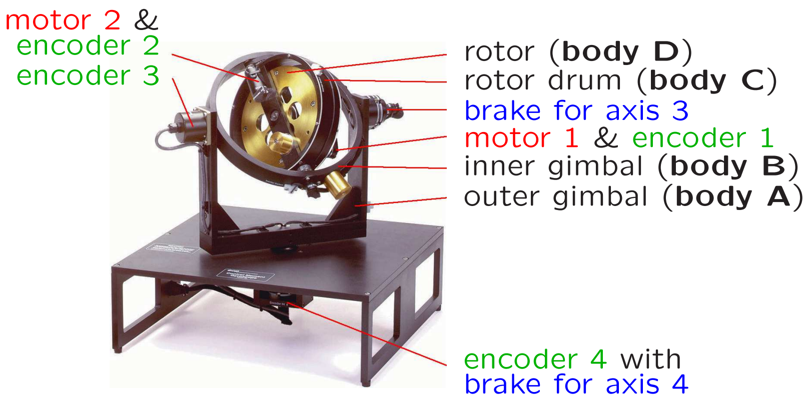


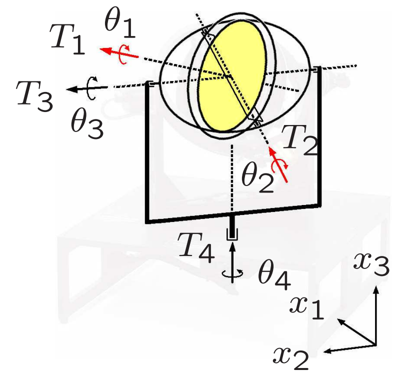
real time controller and host PC

real-time controller: To implement control algorithm and perform digital signal processing. Contains ADC, DSP, DAC & amplifiers.

host-PC: To interact with DSP (run ECP software) and MatLab software.

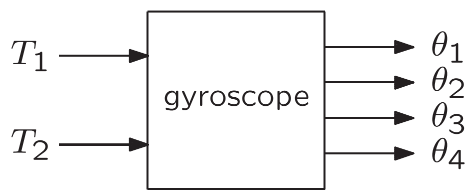
Background theory: modeling a moment exchange gyro





T_i = torque applied to bodies (inputs).

θ_i = angular position of bodies measured by encoders (outputs).



Background theory: modeling a moment exchange gyro

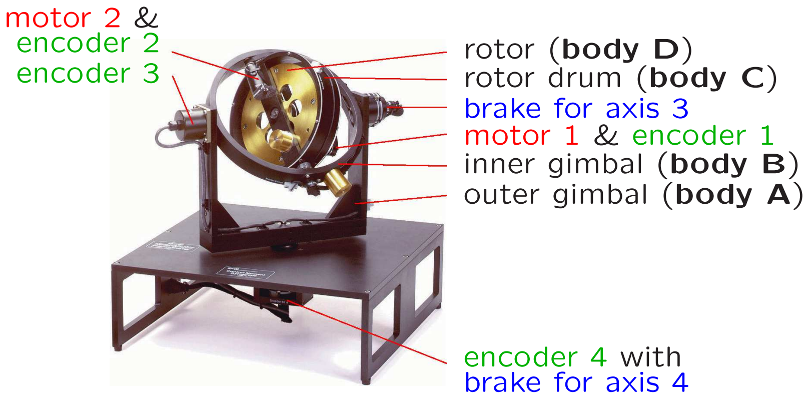
Important parameters (notation) for intertia modeling:

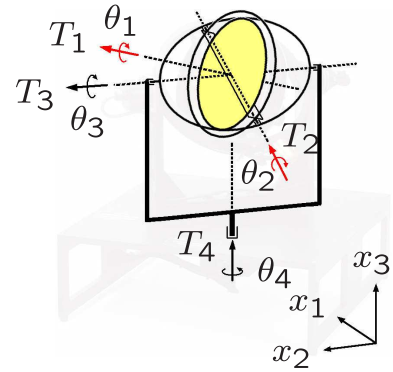
body	definition	angular position	inertia
A	outer gimbal	θ_4	$\mathbf{I}^A$
B	inner gimbal	θ_3	$\mathbf{I}^B$
C	rotor drum	θ_2	$\mathbf{I}^C$
D	rotor	θ_1	$\mathbf{I}^D$

$$\mathbf{I}^b = \begin{bmatrix} I_b & 0 & 0 \\ 0 & J_b & 0 \\ 0 & 0 & K_b \end{bmatrix}, \quad b = \text{body } A, B, C \text{ or } D$$

where I_b denotes the inertia along the x_1-axis, J_b denotes the inertia along the x_2 axis and K_b denotes the inertia along the x_3 axis for $b = A, B, C, D$.

Background theory: modeling a moment exchange gyro





Model found via (non-linear) equations of motion are found by combining:

- Kinematic relations
- Newton's Law ($\sum T = \mathbf{I}\ddot{\theta}$)

Result: relation between angular velocity $\omega^i = \dot{\theta}_i$ and inertia I^i for rotor ($i = 1$), rotor drum ($i = 2$), inner gimbal ($i = 3$) and outer gimbal ($i = 4$)

Background theory: modeling a moment exchange gyro

Non-linear equations of motion are of the form:

$$\begin{aligned}
T_1 &= f_1(\theta_2, \theta_3, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) && (1)\\
T_2 &= f_2(\theta_2, \theta_3, \omega_1, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2) && (2)\\
0 &= f_3(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) && (3)\\
0 &= f_4(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4) && (4)
\end{aligned}$$

and are very hard to work with!

Simplification we will use during experiments in the lab:

Make approximation by considering small perturbations around:

$$\begin{aligned}
\omega_1 &= \Omega && \text{(fixed rotational speed of rotor)}\\
\theta_2 &= \bar{\theta}_2 && \text{(initial position of rotor drum)}\\
\theta_3 &= \bar{\theta}_3 && \text{(initial position of inner gimbal)}
\end{aligned}$$

for model approximation by linearization.

Background theory: modeling a moment exchange gyro

Example of derivation with effects of linearization:

$$J_D\dot{\omega}_1 = T_1 - J_D \cos\bar{\theta}_2\dot{\omega}_3 - J_D \sin\bar{\theta}_2 \cos\bar{\theta}_3\dot{\omega}_4 \quad (1)$$

$$(I_C + I_D)\dot{\omega}_2 = T_2 - J_D\Omega \sin\bar{\theta}_2\omega_3 + J_D\Omega\cos\bar{\theta}_2\cos\bar{\theta}_3\omega_4 + (I_C + I_D)\sin\bar{\theta}_3\dot{\omega}_4 \quad (2)$$

$$(J_B + J_C + J_D - (J_C + J_D - I_D - K_C)\sin^2\bar{\theta}_2)\dot{\omega}_3 =$$
$$-J_D\cos\bar{\theta}_2\dot{\omega}_1 + J_D\Omega\sin\bar{\theta}_2\omega_2 - J_D\Omega\sin\bar{\theta}_2\sin\bar{\theta}_3\omega_4 - \sin\bar{\theta}_2\cos\bar{\theta}_2\cos\bar{\theta}_3 \quad (3)$$

$$(I_B + I_C - K_B - K_C - (J_C + J_D - I_D - K_C)\sin^2\bar{\theta}_2)\sin^2\bar{\theta}_3)\dot{\omega}_4 =$$
$$-J_D\sin\bar{\theta}_2\cos\bar{\theta}_3\dot{\omega}_1 - J_D\Omega\cos\bar{\theta}_2\cos\bar{\theta}_3\omega_2 + (I_C + I_D)\sin\bar{\theta}_3\dot{\omega}_2 +$$
$$J_D\Omega\sin\bar{\theta}_2\sin\bar{\theta}_3\omega_3 - (J_C + J_D - I_D - K_C)\sin\bar{\theta}_2\cos\bar{\theta}_2\cos\bar{\theta}_3\dot{\omega}_3$$
$$-(I_D + K_A + K_B + K_C + (J_C + J_D - I_D - K_C)\sin^2\bar{\theta}_2 \quad (4)$$

Looks complicated, don't expect you to remember this. . .

However, note: *all* red terms are constant! Basically:
4 coupled linear equations that relate $\omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4$ to the torques T_1 and T_2.

Background theory: modeling a moment exchange gyro

The 4 coupled linear equations that relate $\omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4$ to the torques T_1 and T_2 can be written in the matrix notation

$$\mathbf{J}\dot{\omega}(t) = \mathbf{D}\omega(t) + \mathbf{Q}T(t)$$

where the vector $\omega(t)$ of rotational speeds is given by

$$\omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \\ \omega_4(t) \end{bmatrix} \text{ and } T(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}$$

and where $\mathbf{J}$, $\mathbf{D}$ and $\mathbf{Q}$ are all previous red terms based on a linearization around

$$\begin{aligned} \omega_1 &= \Omega \quad \text{(fixed rotational speed of rotor)} \\ \theta_2 &= \bar{\theta}_2 \quad \text{(initial position of rotor drum)} \\ \theta_3 &= \bar{\theta}_3 \quad \text{(initial position of inner gimbal)} \end{aligned}$$

Background theory: modeling a moment exchange gyro

Choose:
$$\begin{array}{rcll} \omega_1 &=& \Omega & \text{(constant rotor speed)} \\ \theta_2 &=& \bar{\theta}_2 = 0 & \text{(rotor drum perpendicular)} \\ \theta_3 &=& \bar{\theta}_3 = 0 & \text{(inner gimbal perpendicular)} \end{array}$$

then the 4 coupled linear differential equations reduce to

$$\begin{array}{rcl} J_D\dot{\omega}_1 + J_D\dot{\omega}_3 &=& T_1 \\ (I_C + I_D)\dot{\omega}_2 &=& J_D\Omega\omega_4 + T_2 \\ (J_B + J_C + J_D)\dot{\omega}_3 + J_D\dot{\omega}_1 &=& 0 \\ (I_D + K_A + K_B + K_C)\dot{\omega}_4 &=& -J_D\Omega\omega_2 \end{array}$$

and we see $\mathbf{J}\dot{\omega}(t) = \mathbf{D}\omega(t) + \mathbf{Q}T(t)$ with

$$\mathbf{J} = \begin{bmatrix} J_D & 0 & J_D & 0 \\ 0 & (I_C + I_D) & 0 & 0 \\ J_D & 0 & (J_B + J_C + J_D) & 0 \\ 0 & 0 & 0 & (I_D + K_A + K_B + K_C) \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_D\Omega \\ 0 & 0 & 0 & 0 \\ 0 & -J_D\Omega & 0 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Background theory: modeling a moment exchange gyro

If we extend the vector $\omega(t)$ of angular velocities to also contain (some of) the angular positions

$$x(t) = \left[\ \theta_2(t) \quad \theta_3(t) \quad \theta_4(t) \quad \omega_1(t) \quad \omega_2(t) \quad \omega_3(t) \quad \omega_4(t)\ \right]^T$$

then we can write down a so-called *state space model*

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu(t) \\ y(t) &=& Cx(t) \end{array}$$

where $u(t)$ are inputs (torques by motor 1 or motor 2) and $y(t)$ are outputs (angular rotation from encoders).
In our case matrix C will be

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for } \theta_2,\ \theta_3 \text{ and } \theta_4 \text{ as output}$$

With $\dot{\theta}_i(t) = \omega_i(t)$ and $\mathbf{J}\dot{\omega}(t) = \mathbf{D}\omega(t) + \mathbf{Q}T(t)$ we see

$$A = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{U}_{3\times 4} \\ \mathbf{0}_{4\times 4} & \mathbf{J}^{-1}\mathbf{D} \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0}_{3\times 3} \\ \mathbf{J}^{-1}\mathbf{Q} \end{bmatrix}, \quad \mathbf{U}_{3\times 4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Background theory: modeling a moment exchange gyro

Format of linearized model of gyro in state space form:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array}, \text{ where}$$

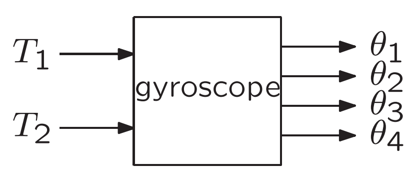
$$u(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}, \; y(t) = \begin{bmatrix} \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \end{bmatrix},$$

$$x(t) = \begin{bmatrix} \theta_2(t) & \theta_3(t) & \theta_4(t) & \omega_1(t) & \omega_2(t) & \omega_3(t) & \omega_4(t) \end{bmatrix}^T \text{ and}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{J_D\Omega}{I_C+I_D} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-J_D\Omega}{I_D+K_A+K_B+K_C} & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{J_B+J_C+J_D}{J_D(J_B+J_C)} & 0 \\ 0 & \frac{1}{I_C+I_D} \\ \frac{-1}{J_B+J_C} & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Background theory: modeling a moment exchange gyro



Resulting model of gyro is a linear and multivariable model.

Further assumptions for your laboratory experiment:

- Voltage applied to motor 1 & motor 2 can be used to create external torque T_1 and T_2.
- Angular positions θ_i = angular position for $i = 2, 3, 4$ can be measured via angular encoders.

Further simplifications for your laboratory experiment:

- Lock one or more axis using brake switches for axis 3 or 4 and virtual brake for axis 2.
- Consider only ONE motor to create torgue T_j and ONE measurement θ_i at the time for simplified Single Input Single Output (SISO) control.

Background theory: modeling a moment exchange gyro

Impact of this simplification:

- Using only ONE motor to create a torgue T_j and ONE measurement θ_i at the time, reduces the multivariable model to a different SISO models.
- This SISO model can be written as a transfer function $G_{ij}(s)$

MAIN RESULT

Let C_i be the ith row of the matrix C and let B_j denotes the jth column of the matrix B, then

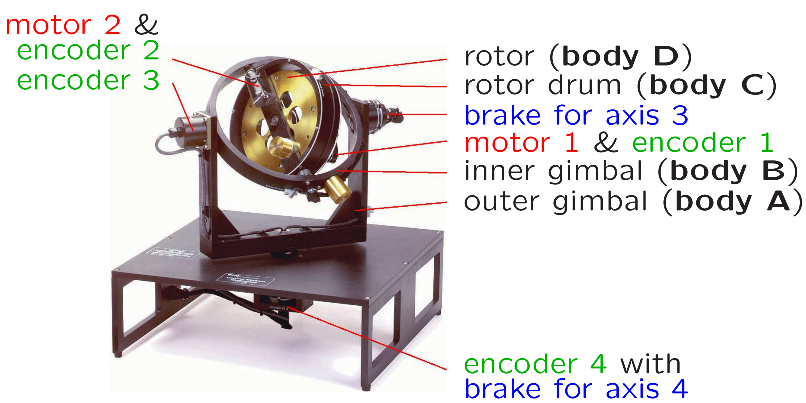
$$G_{ij}(s) = C_i(sI - A)^{-1}B_j$$

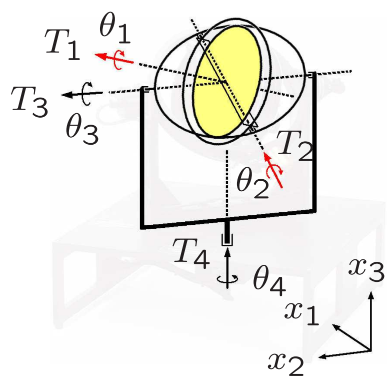
where $i = 2, 3, 4$ due to encoder 2, 3 or 4 and $i = 1, 2$ due to motor input 1 or 2.

Example: transfer function from motor 2 to encoder 4

$$\theta_4(s) = G_{42}(s)T_2(s), \quad G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} \cdot \frac{1}{s}$$

Background theory: modeling a moment exchange gyro





Configurations of the simplified gyroscope during your lab:

- Week 1: brake 4 on and (virtual) brake 2 on.
  Measure θ_3, control motor 1: inertial control.
- Week 2: brake 3 on and $\omega_1 = \Omega$.
  Measure θ_2, control motor 2: nutation frequency control.
- Week 3: brake 3 on and $\omega_1 = \Omega$.
  Measure θ_4, control motor 2: precession control.

Background theory: modeling a moment exchange gyro

Models during your lab:

- Week 1: brake 4 on and (virtual) brake 2 on leads to

$$y_3(s) = G_{31}(s)u_1(s), \text{ with } G_{31}(s) = \frac{-1}{(J_B + J_C)s^2}$$

- Week 2: brake 3 on and $\omega_1 = \Omega$: $y_2(s) = G_{22}(s)u_2(s)$ with

$$G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2}$$

- Week 3: brake 3 on and $\omega_1 = \Omega$: $y_4(s) = G_{42}(s)u_2(s)$ with

$$G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s}$$

Background theory: reduction to 1DOF system

Transfer functions $G_{31}(s)$, $G_{22}(s)$ and $G_{42}(s)$ look complicated, but are basically of the form:

$$\begin{aligned}
G_{31}(s) &= \frac{-1}{(J_B + J_C)s^2} &= \frac{K_0}{s^2 + \beta_0 s} \\
G_{22}(s) &= \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} &= \frac{K_1 {\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2} \\
G_{42}(s) &= \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} &= \frac{K_2 \omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{1}{s}
\end{aligned}$$

where we have the (lumped) parameters:

- scaling K_0, (additional) damping β_0
- scaling K_1, damping β and nutation frequency ω_n
- scaling K_2, damping β and nutation frequency ω_n

Background theory: reduction to 1DOF system

To be able to do control design, you need to find the parameters of the transfer functions

$$\begin{aligned} G_{31}(s) &= \frac{-1}{(J_B + J_C)s^2} &&= \frac{K_0}{s^2 + \beta_0 s} \\ G_{22}(s) &= \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} &&= \frac{K_1 {\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2} \\ G_{42}(s) &= \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} &&= \frac{K_2 \omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \end{aligned}$$

Hence, for modeling we would need to determine inertia of bodies in three different axes. . .

Alternative to determining inertia of bodies in three different axes: determine (lumped) parameters K_0, K_1, K_2, β_0, β_1 and ω_n via **dynamic experiments** (step response experiments).

Background theory: step response of a 1st order system

Let us consider

$$G_{31}(s) = \frac{K_0}{s^2 + \beta_0 s}$$

as our model for our 1st week laboratory experiment:

- **Instead** of using the *position* $\theta_3(t)$ as output, consider using the *velocity* $\omega_3(t)$ as output.
- With $\omega_3(t) = \frac{d}{dt}\theta_3(t)$ we have $\omega_3(s) = s\theta_3(s)$ using Laplace.
- **Hence**, choosing $\omega_3(t)$ as an output $y_3(t)$ modifies $G_{31}(s) \cdot s$ to a simple 1st order system:

$$y_3(s) = \frac{K_0}{s + \beta_0} u_1(s)$$

You can determine (K_0, β_0) from a dynamic (step response) experiment. . .

Background theory: step response of a 1st order system

MAIN RESULT

Consider a 1st order system with a transfer function

$$G_{ij}(s) = \frac{K}{s+\beta}$$

then a step input $u(t) = U,\ t \geq 0$ of size U results in the output response

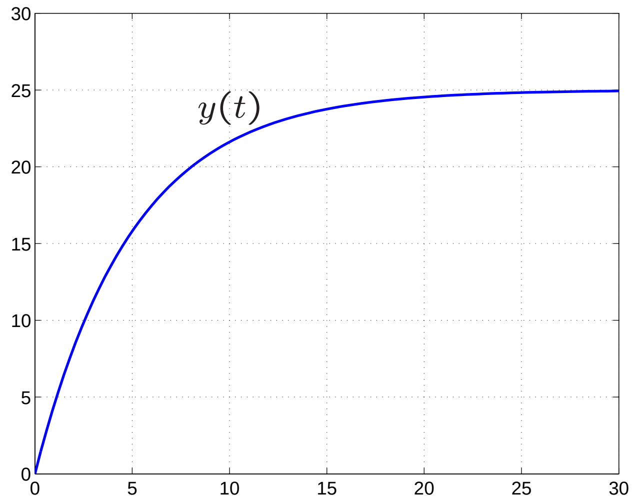
$$y(t) = \frac{K}{\beta} \cdot U \cdot \left[1 - \mathrm{e}^{-\beta t}\right]$$

Result follows directly from inverse Laplace transform of

$$y(s) = G_{ij}(s)\frac{U}{s} = \frac{KU}{s+\beta} \cdot \frac{1}{s}$$

Background theory: step response of a 1st order system

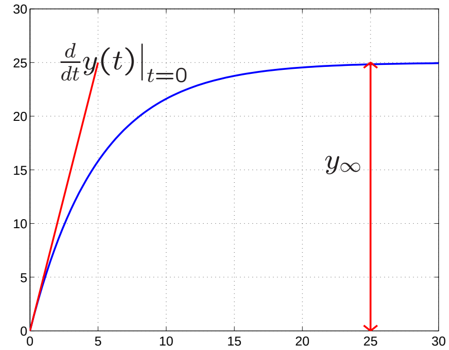
Typical picture of $y(t) = \frac{K}{\beta} \cdot U \cdot \left[1 - \mathrm{e}^{-\beta t}\right]$ for a step size of $U = 1$, scaling $K = 5$, and (inverse) time constant $\beta = 0.2$:



How about the inverse problem of finding β and K from (velocity) measurement $y(t)$?

Background theory: step response of a 1st order system

With $\frac{d}{dt}y(t)\Big|_{t=0}$ and y_∞ from velocity step response:



We see $\frac{d}{dt}y(t)\Big|_{t=0} = K \cdot U e^{-\beta t}\Big|_{t=0} = K \cdot U$ and hence

$$\hat{K} = \frac{1}{U} \cdot \frac{d}{dt}y(t)\Big|_{t=0}$$

Knowing that the (velocity) $\lim_{t\to\infty} y(t) = y_\infty = \frac{K}{\beta} \cdot U$ we find

$$\hat{\beta} = \frac{\hat{K}}{y_\infty} \cdot U$$

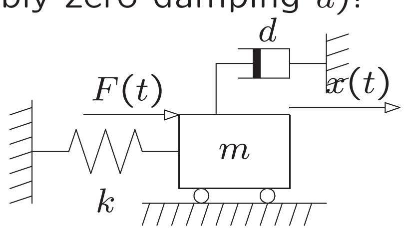
Background theory: step response of a 2nd order system

Let us consider

$$G_{22}(s) = \frac{K_1 {\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2}$$

as our model for our 2nd week laboratory experiment:

- This is a standard second order system, where ω_n = undamped (nutation) frequency, $0 \leq \beta \leq 1$ is the damping ratio, K is the gain.
- This is similar to the transfer function of a 1DOF mechanical system (with possibly zero damping d)!



$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = F(t) \;\Rightarrow\; x(s) = \frac{1}{ms^2 + ds + k} F(s)$$

Background theory: step response of a 2nd order system

In case $k > 0$ we can write

$$G(s) = \frac{1}{ms^2 + ds + k} = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}$$

where

$$K := \frac{1}{k} \text{ (compliance)},$$
$$\omega_n := \sqrt{\frac{k}{m}} \text{ (resonance)},$$
$$\beta := \frac{1}{2}\frac{d}{\sqrt{mk}} \text{ (damping ratio)}$$

Note the resemblance with the 2nd week lab experiment model

$$G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} = \frac{K_1 {\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2}$$

You you determine (K_1, ω_n, β) from a dynamic (step response) experiment. . .

Background theory: step response of a 2nd order system

MAIN RESULT

Consider a 2nd order system with a transfer function

$$G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}$$

then a step input $u(t) = U, \ t \geq 0$ of size U results in the output response

$$y(t) = K \cdot U \cdot \left[1 - \mathrm{e}^{-\beta\omega_n t}\sin(\omega_d t + \phi)\right]$$

where

$$\omega_d = \omega_n\sqrt{1-\beta^2} \quad \text{damped resonance frequency in rad/s}$$
$$\phi = \tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta} \quad \text{phase shift of response in rad}$$

Background theory: step response of a 2nd order system

DERIVATION for 1DOF (second order) system
Compute the dynamic response via inverse Laplace transform!

Consider a step input $u(t) = U,\ t \geq 0$ of size U. Then $u(s) = \frac{U}{s}$ and for the 1DOF system we have

$$y(s) = G(s)u(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{U}{s}$$

and the inverse Laplace transform is given by

$$y(t) = K \cdot U \cdot \left[1 - \mathrm{e}^{-\beta\omega_n t} \sin(\omega_d t + \phi)\right]$$

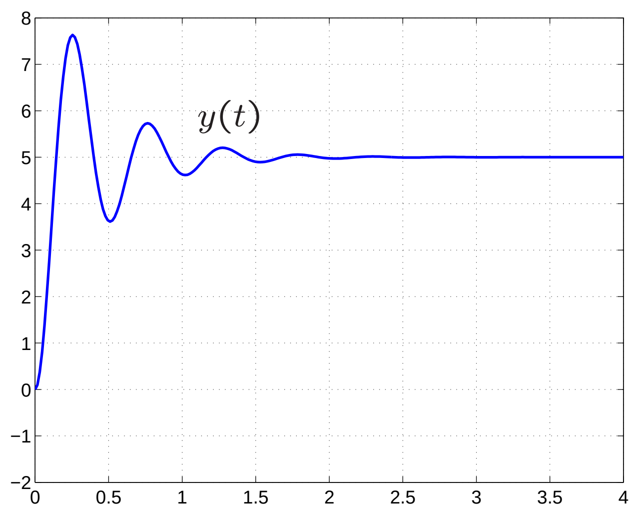
where

$$\omega_d = \omega_n\sqrt{1-\beta^2} \quad \text{damped resonance frequency in rad/s}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta} \quad \text{phase shift of response in rad}$$

Background theory: step response of a 2nd order system

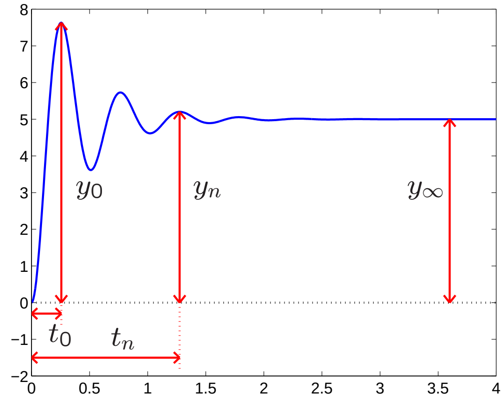
Typical picture of $y(t) = K \cdot U \left[1 - \mathrm{e}^{-\beta\omega_n t} \sin(\omega_d t + \phi)\right]$ for a step size of $U = 1$, scaling $K = 5$, undamped resonance frequency $\omega_n = 2 \cdot 2\pi \approx 12.566$ rad/s and damping ratio $\beta = 0.2$:



How about the reverse problem of finding ω_n, β and K from $y(t)$?

Background theory: step response of a 2nd order system

With the times t_0, t_n and the values y_0, y_n and y_∞ from (position or velocity) step response:



Allows us to estimate:

$$\widehat{\omega}_d = 2\pi \frac{n}{t_n - t_0} \quad \text{(damped resonance frequency)}$$

$$\widehat{\beta\omega_n} = \frac{1}{t_n - t_0} \ln\left(\frac{y_0 - y_\infty}{y_n - y_\infty}\right) \quad \text{(exponential decay term)}$$

where $n =$ number of oscillations between t_n and t_0.

Background theory: step response of a 2nd order system

With the estimates

$$\widehat{\omega}_d = 2\pi \frac{n}{t_n - t_0} \quad \text{(damped resonance frequency)}$$

$$\widehat{\beta\omega_n} = \frac{1}{t_n - t_0} \ln\left(\frac{y_0 - y_\infty}{y_n - y_\infty}\right) \quad \text{(exponential decay term)}$$

we can now compute $\widehat{\omega}_n$ and $\widehat{\beta}$:

$$\widehat{\omega}_n = \sqrt{\widehat{\omega}_d^2 + (\widehat{\beta\omega_n})^2} \quad \text{(undamped resonance frequency)}$$

$$\widehat{\beta} = \frac{\widehat{\beta\omega_n}}{\widehat{\omega}_n} \quad \text{(damping ratio)}$$

and K follows from

$$\widehat{K} = \frac{y_\infty}{U} \quad \text{(compliance)}$$

Background theory: step response of a 2nd order system

Let us now consider

$$G_{42}(s) = \frac{K_2\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

as our model for our 3rd week laboratory experiment:

- **Instead** of using the *position* $\theta_4(t)$ as output, consider using the *velocity* $\omega_4(t)$ as output.
- With $\omega_4(t) = \frac{d}{dt}\theta_4(t)$ we have $\omega_4(s) = s\theta_4(s)$ using Laplace.
- **Hence**, choosing $\omega_4(t)$ as an output $y_4(t)$ modifies $G_{42}(s) \cdot s$ back into a standard 2nd order system

$$y_4(s) = \frac{K_2\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} u_2(s)$$

NOTE: we can use the same step response results and parameter estimation as we did in our 2nd week laboratory experiment to estimate K_2, β and ω_n.

Outline of Lab Work: estimation of model parameters

Important observation: gyroscope in 1st week lab experiment appears (dynamically) equivalent to a second order system with

$$G_{ij}(s) = \frac{K_0}{s^2 + \beta_0 s}$$

and can be reduced to a 1st order system

$$G_{ij}(s) \cdot s = \frac{K_0}{s + \beta_0}$$

when using the *velocity* $\omega_3(t)$ as output $y_3(t)$.

Main Idea:

- Experiment: use a step response *laboratory experiment* to formulate parameter estimates $\widehat{K}$ and $\widehat{\beta}_0$.
- Validate model, so that measured step response looks similar to simulated step response

Outline of Lab Work: estimation of model parameters

Important observation: gyroscope in 2nd and 3rd week lab experiment appears (dynamically) equivalent to a single 1DOF with a transfer function

$$G_{ij}(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}$$

where

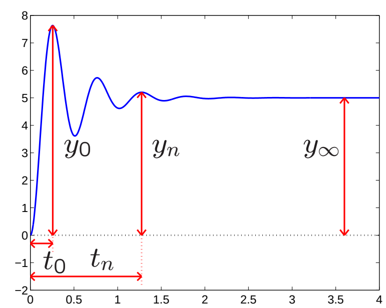
$$\begin{aligned} K &:= \text{scaling}, \\ \omega_n &:= \text{nutation frequency}, \\ \beta &:= \text{damping ratio} \end{aligned}$$

Main Idea:

- Experiment: use a step response *laboratory experiment* to formulate parameter estimates $\widehat{K}$, $\widehat{\omega}_n$ and $\widehat{\beta}$.
- Validate model, so that measured step response looks similar to simulated step response

Outline of Lab Work: estimation of model parameters

EXAMPLE: step of $U = 0.5V$ on motor



Read from plot:

$t_0 = 0.25,\ t_n = 1.25,\ y_0 = 7.5,\ y_n = 5.25,\ y_\infty = 5$

$$\begin{aligned} \widehat{\omega}_d &= 2\pi\frac{n}{t_n - t_0} = 2\pi\frac{2}{1.25 - 0.25} = 4\pi \\ \widehat{\beta\omega_n} &= \frac{1}{t_n - t_0}\ln\left(\frac{y_0 - y_\infty}{y_n - y_\infty}\right) = \frac{1}{1.25 - 0.25}\ln\left(\frac{7.5 - 5}{5.25 - 5}\right) = \ln(10) \\ \widehat{\omega}_n &= \sqrt{\widehat{\omega}_d^2 + (\widehat{\beta\omega_n})^2} = \sqrt{16\pi^2 + \ln(10)^2} \approx 12.78 \\ \widehat{\beta} &= \frac{\widehat{\beta\omega_n}}{\widehat{\omega}_n} \approx \frac{\ln(10)}{12.78} \approx 0.18 \\ \widehat{K} &= \frac{y_\infty}{U} = \frac{5}{0.5} = 10 \end{aligned}$$

Units of $\widehat{K}$? Does it matter?

Outline of Lab Work: estimation of model parameters

Conclusions and summary of work:

- Verify transfer function you are working with (week 1,2 or 3)
- It will be of the form

$$G_{ij}(s) = \frac{K}{s+\beta} \cdot \left(\cdot\frac{1}{s}\right) \quad \text{or} \quad G_{ij}(s) = \frac{K{\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2}\left(\cdot\frac{1}{s}\right)$$

- Estimate the parameters model parameters (K, ω_n, β) via the measurement of a step response.
- Verify the simulation of your step response with the measurement of a step response.
- Substitute parameter values to create your dynamic model $G(s)$ to be used for control design

Luckely, we have Matlab and a matlab script/function called `maelab.m` to help you with his.

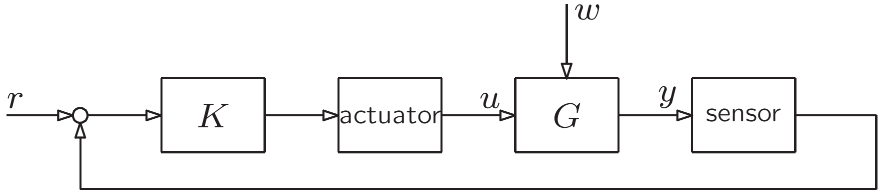
Simply enter your models in a file called `models.m`

Outline of Lab Work: estimation of model parameters

NOTES:

- During week 1 you DO NOT use a constant rotor speed Ω, as you will be controlling the rotor motor 1 yourself.
- During week 2 & 3 you DO use a constant rotor speed Ω.
- For each laboratory section a different constant rotational rotor speed Ω is used.
- The rotational rotor speed you will be working with is given in the `config.txt` or `info.txt` file.
- Model parameters (K, ω_n, β) will depend on physical quantities (mass, inertia, damping, etc.) and need to be determined during laboratory
- Important in validation of laboratory work: comparison of actual (dynamic) experiments with a (dynamic) simulation of your model $G(s)$ before you start doing control design!
- With a bad model you cannot do a proper control design. . .

Outline of Lab Work: model validation



Keep in mind:

- From parameter estimation experiments we know obtain a model specified as a transfer function model $y(s) = G_{ij}(s)u(s)$

$$G_{ij}(s) = \frac{K}{s^2 + \beta s} \quad \text{or} \quad G_{ij}(s) = \frac{K{\omega_n}^2}{s^2 + 2\beta\omega_n s + {\omega_n}^2}\left(\cdot\frac{1}{s}\right)$$

  where $u(s) =$ is control input (motor Voltage or Force) and $y(t) =$ system output (position in encoder counts)
- Model $G(s)$ is created automatically for you via `maelab.m` script file by modifying the `models.m` file.
- You are going to use the model $G(s)$ to design a controller $K(s)$, so model $G(s)$ should be validated first!

Outline of Lab Work: model validation

What is suitable for model validation:

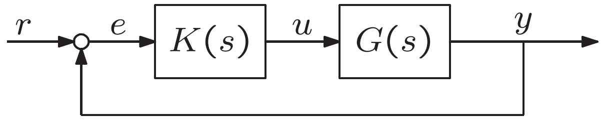
- Validate the estimation of each set of model parameters by comparing actual experiments with a simulation of a 1DOF model $G_{ij}(s)$.

  This can be done with the `maelab.m` script file.

- With a bad (unvalidated) model $G_{ij}(s)$ you cannot do a proper model-based control $K(s)$ design!

Outline of Lab Work: design of controllers

Each week you construct feedback loop around $G(s)$:



Schematic view of closed loop configuration

Find a feedback controller $K(s)$ that satisfies:

- move a mass/inertia to a certain (angular) position as fast as possible
- limit overshoot during control/positioning to 25%
- no steady-state error e
- illustrate disturbance rejection when control is implemented

Trade off in design specifications:

high speed $\leftrightarrow$ overshoot
overshoot $\leftrightarrow$ robustness

Outline of Lab Work: design of controllers

Controller configurations that can be implemented:

- P-control

$$u(t) = k_p e(t),\ e(t) = r(t) - y(t)$$

- PD-control

$$u(t) = k_p e(t) + k_d \frac{d}{dt} e(t),\ e(t) = r(t) - y(t)$$

- PID control

$$u(t) = k_p e(t) + k_i \int_{\tau=0}^{t} e(\tau) d\tau + k_d \frac{d}{dt} e(t),\ e(t) = r(t) - y(t)$$

Outline of Lab Work: design of controllers

All controllers are implemented in a discrete-time!

Let T_s = sample time, $t = kT_s$ and $e(k) = r(k) - y(k)$ we get

- P-control

$$u(k) = k_p e(k)$$

- PD-control

$$u(k) = k_p e(k) + \frac{k_d}{T_s}[e(k) - e(k-1)]$$

- PID control

$$u(k) = k_p e(k) + k_i T_s \sum_{l=0}^{k} e(l) + \frac{k_d}{T_s}[e(k) - e(k-1)]$$

Templates will be available for you to enter your control parameters k_p, k_d and/or k_i.

Outline of Lab Work: design of controllers

Example of template

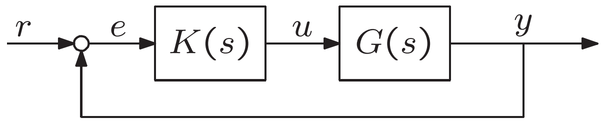
```
;***********define  user variables **************
#define kp q2        ; proportional gain
#define kd q3        ; derivative gain
#define ki q4        ; integral gain
#define Ts q5        ; sampling time for scaling of derivative and integral gain
#define kdd q6       ; scaled derivative gain
#define kii q7       ; scaled integral gain
#define enc2_prev q8 ; keep track of previous encoder 2 measurement for derivative
#define error q9     ; keep track of error signal for integration

;************Initialize  variables****************
Ts=.00442   ; For local use only.  Must ALSO be set in SETUP CONTROL ALGORITHM dialog box!
kp=0.2      ; set proportional gain
kd=0.02     ; set derivative gain
ki=0.2      ; set integral gain
kdd=kd/Ts   ; divide by Ts here to save real-time processing
kii=ki*Ts   ; multiply by Ts here to save real-time processing
error=0     ; initialize integrator error

;PID Controller that uses encoder 2 and motor 2 (nutation control)
;********* real time code which is run every servo period ***
begin
 error=error+cmd2_pos-enc2_pos
 control_effort2=kp*(cmd2_pos-enc2_pos)+kii*error+kdd*(enc2_prev-enc2_pos)
 enc2_prev=enc2_pos
end
```

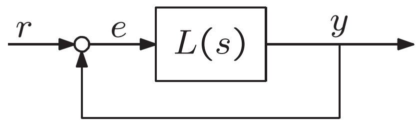
Outline of Lab Work: design of controllers (loop gain)

Model $G(s)$ of plant should be used for design of controller $K(s)$!



For design of controller $K(s)$, consider the loop gain:

$$L(s) := K(s)G(s)$$



loop gain: series connection of $K(s)$ and $G(s)$

Dynamics of loop gain $L(s)$ consists of fixed part $G(s)$ (plant dynamics) and to-be-designed part $K(s)$ (controller)

Outline of Lab Work: design of controllers (stability)

Loop gain $L(s) := K(s)G(s)$ important for:

- Stability
- Design specification

STABILITY:
With closed-loop poles found by those values of $s \in \mathbb{C}$ that satisfy

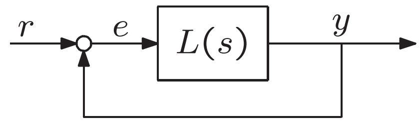
$$1 + L(s) = 0$$

all closed-loop poles should have negative real values (lie in the left part of the complex plane)

Stability can be checked by:

- Actually computing the solutions to $L(s) = -1$ as a function of k_p, k_i, k_d: Root Locus Method
- See if Nyquist plot of $L(s)$ encircles the point -1 as a function of k_p, k_i, k_d: Nyquist or Frequency Domain Method

Outline of Lab Work: design of controllers (design specs)



Error rejection transfer function:

$$E(s) = \frac{1}{1 + L(s)} \quad \text{(map from } r \text{ to } e\text{)}$$

To avoid a steady state error $e(t)$ as $t \to \infty$, one specification for the loop gain can be found via the final value theorem. With $L(s) = G(s)K(s)$ we have:

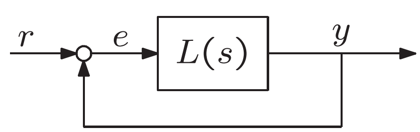
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot \frac{1}{1 + L(s)} r(s)$$

With $r(t) =$ (unit) step input, $r(s) = \frac{1}{s}$ and

$$\lim_{s\to 0} |L(s)| = \infty$$

is needed for zero steady-state behavior!

Outline of Lab Work: design of controllers (design specs)



Closed loop transfer function:

$$T(s) = \frac{L(s)}{1 + L(s)} \quad \text{(map from } r \text{ to } y\text{)}$$

To make sure y follows r, we would like to make $T(s) = 1$ as close as possible.

Notice that with Error rejection transfer function:

$$E(s) = \frac{1}{1 + L(s)} \quad \text{(map from } r \text{ to } e\text{)}$$

we have

$$T(s) + E(s) = 1$$

Hence, if you can make $|E(s)| \approx 0$ small, then $|T(s)| \approx 1$.

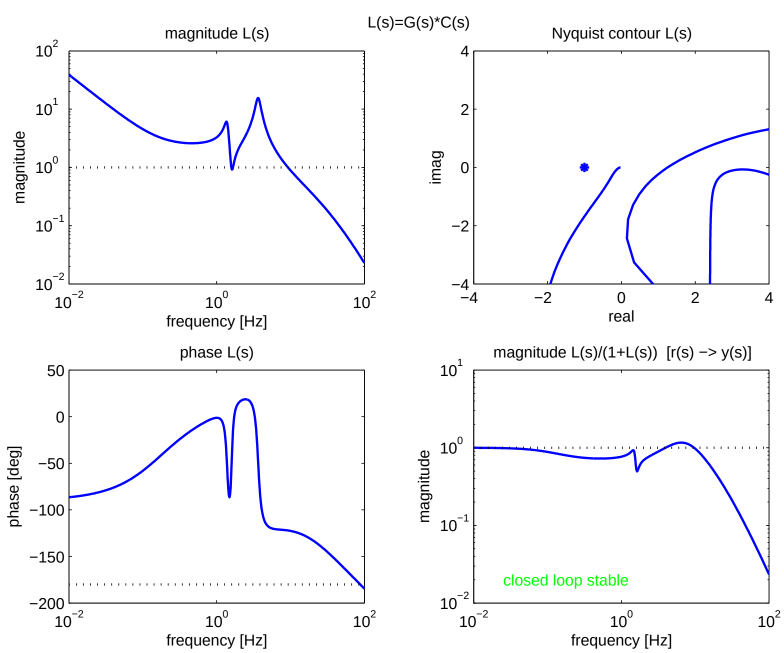
Outline of Lab Work: design of controllers (graphical design)

Computation of $K(s)$: translate design specifications to $L(s)$, $E(S)$ or $T(s)$ specifications.

- Use graphical analysis and design utilities (root locus or frequency domain methods) to shape loop gain $L(s)$ and design controller $K(s)$.
- Root-locus and frequency domain design method is implemented in Matlab via the `rltool` command.
- Frequency domain design method has also been implemented in a Matlab script file `maelab` provided during the lab.
- Stability via Nyquist criterion (do not encircle point -1)
- Phase and amplitude margin (stability and robustness) translate to shape Bode plot of loop gain $L(s) = G(s)K(s)$:
  phase margin: when $|L(s)| = 1, \angle L(s) > -\pi$ rad
  amplitude margin: when $\angle L(s) = -\pi$ rad, $|L(s)| < 1$

Outline of Lab Work: design of controllers (graphical design)

Example of figures produced by `maelab` script file



Outline of Lab Work: design of controllers (general trend)

Effects of control parameters: for PD-control and a standard 2nd order plant model this can be analyzed as follows:

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{(k_p + k_d s)\frac{\omega_n^2}{s^2+2\beta\omega_n s+\omega_n^2}}{1 + (k_p + k_d s)\frac{\omega_n^2}{s^2+2\beta\omega_n s+\omega_n^2}}$$

which yields the closed-loop transfer function

$$T(s) = \frac{\omega_n^2(k_p + k_d s)}{s^2 + 2\bar{\beta}\bar{\omega}_n + \bar{\omega}_n^2} \text{ with } \bar{\omega}_n = \omega_n\sqrt{1+k_p} \text{ and } \bar{\beta} = \frac{\beta + \omega_n k_d/2}{\sqrt{1+k_p}}$$

In this case $T(s)$ is also a second order system and with knowledge of the step response, we can conclude that the following influence of the controller parameters:

- $k_p \leftrightarrow$ speed of response
- $k_p \leftrightarrow$ damping
- $k_p \leftrightarrow$ steady-state error
- $k_d \leftrightarrow$ damping

Outline of Lab Work: design of controllers (summary)

- Model $G(s)$ of plant should be used for design of your controller $K(s)$!
- Think about the controller needed (P, PD or PID) to stabilize your system
- Obviously, controllers $K(s)$ for week 1, 2 and 3 are all different and $G(s)$ is different!
- Keep in mind the requirement of 25% overshoot, and no steady state error, e.g. $r(t) = y(t)$ as $t \to \infty$.
- Use graphical design tools to design your P, PD and PID control:
  - Root-locus and frequency domain design method is implemented in Matlab via the `rltool` command.
  - Frequency domain design method also implemented in a Matlab script file `maelab` provided during the lab.

Outline of Lab Work: design of controllers (summary)

- Consider how the frequency resp of P- and PD- and PID controller modifies the loop gain $L(s) = G(s)K(s)$. Look at asymptotes of Bode plot of controller

$$K(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

- Phase and amplitude margin (stability and robustness) translate to shape Bode plot of loop gain $L(s) = G(s)K(s)$:
  phase margin: when $|L(s)| = 1, \angle L(s) > -\pi$ rad
  amplitude margin: when $\angle L(s) = -\pi$ rad, $|L(s)| < 1$
- Argument and motive your control design in rapport.
- No trial-and-error control design results are accepted.

Summary of Lab Work

- *first week*
  Study laboratory handout. Get familiar with the multi-input-multi-output (MIMO) moment gyroscope system, introduction to ECP software used for experiments and controller implementation. Engage axis 4 brake (lock outer gimbal) and virtual brake 2 (lock rotor drum). Design and implement feedback using motor 1 as input and encoder 3 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Design controller based on model in Matlab and implement/verify on actual system.
- *second week*
  Engage axis 3 brake (lock inner gimbal). Bring rotor at fixed speed Ω. Design and implement feedback using motor 2 as input and encoder 2 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Validation of model parameters via comparison of simulation and experiments.Design controller based on model in Matlab and implement/verify on actual system.
- *third week*
  Engage axis 3 brake (lock inner gimbal). Bring rotor at fixed speed Ω. Design and implement feedback using motor 2 as input and encoder 4 as output. Propose experiments to estimate (unknown) parameters in your model of the system. Validation of model parameters via comparison of simulation and experiments. Validation of nutation & precession of MIMO moment gyroscope dynamics.Design controller based on model in Matlab and implement/verify on actual system.

What should be in your report (1-2)

- Abstract
  Standalone - make sure it contains clear statements w.r.t motivation, purpose of experiment, main findings (numerical) and conclusions.
- Introduction
  - Motivation (why are you doing this experiment)
  - Short description of the main engineering discipline (controls)
  - Answer the question: what is the aim of this experiment/report?
- Theory
  - Feedback system
  - Modeling
  - Parameter estimation
  - Control design

What should be in your report (2-2)

- Experimental Procedure
  - Short description of experiment
  - How are experiments done (detailed enough so someone else could repeat them)
- Results
  - Estimation of parameters K, ω_n and β
  - Validation of models $G_{ij}(s)$
  - Design of controller $K(s)$ and Implementation
- Discussion
  - Why are simulation results different from experiments?
  - Could the model be validated?
  - Are designed controller parameters O.K. from model?
- Conclusions
- Error Analysis
  - Mean, standard deviation and 99% confidence intervals of estimated parameters ω_n, β and K from data