Vibration Analysis Experiment – MAE175A

Flexible Building and Helicopter Rotor Blade

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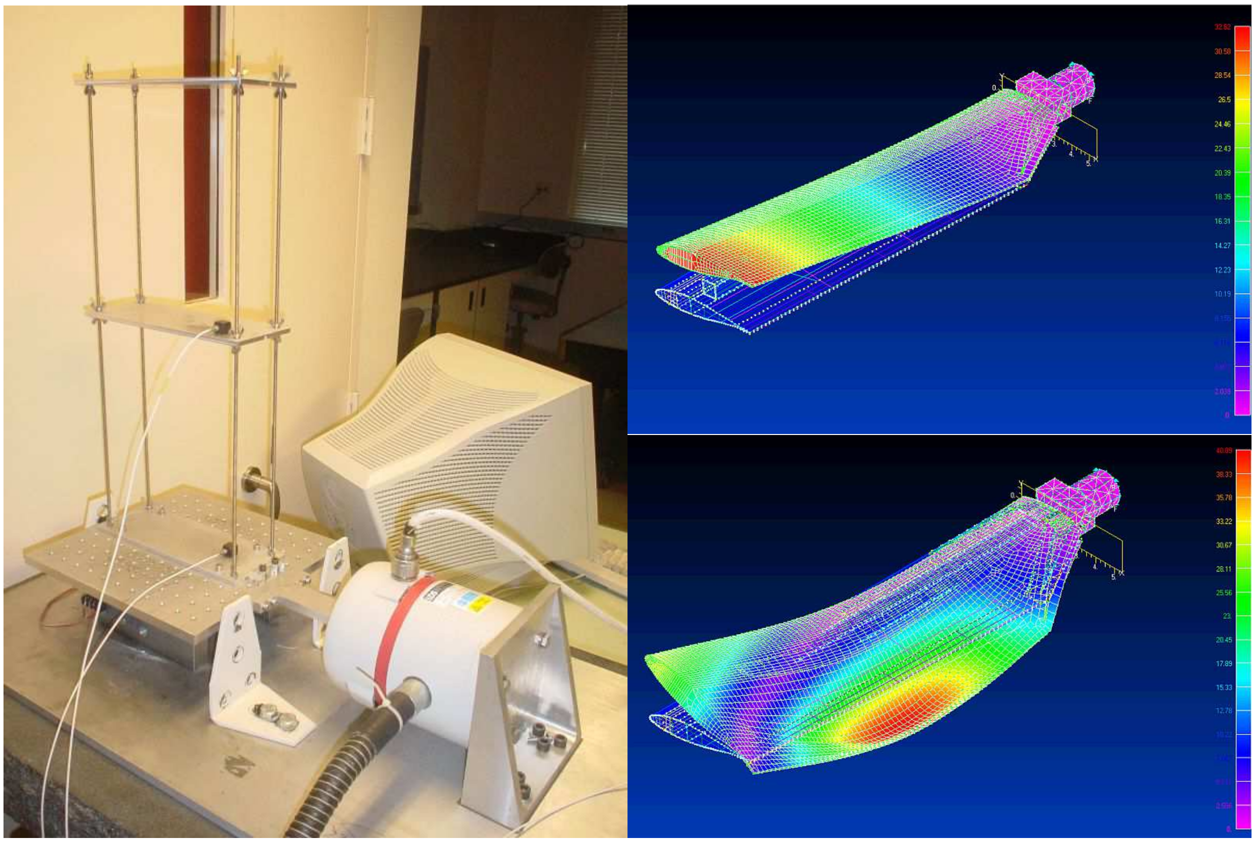

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1.1 Aim of experiment

The vibration analysis experiment consists of lightly damped scaled down three story building and a helicopter rotor blade on a shaker. The scaled down flexible structure consists of three stories connected by a flexible elements and the shaker is used to simulate the behavior of a flexible building subjected to base force and acceleration disturbances similar to earthquake excitation. The helicopter rotor blade is mounted to the optical table at one end and actuation is applied at the other end. The mechanical structures are equipped with accelerometers to study and experimentally verify resonance frequencies, mode shapes and frequency responses.

> **The aim of this vibration analysis experiment is to understand and measure phenomena associated to resonance frequencies and mode shapes of a flexible structure and formulate models that can validate a measured frequency responses.**

In this experiment the understanding of measurable phenomena associated to the flexible three-story building is provided by an analytic model that is derived by means of Lagrange's method. The model of the helicopter rotor blade is given to you in the form of a developed Finite Element Analysis (FEA) model that predicts the particular mode shapes. The dynamical properties of the vibration model should be verified and validated on the basis of models and experimental data obtained from the structure.

1.2 Laboratory procedure

The lab course will consists of 3 laboratory sessions of 3 hours each. There are two accelerometers that can be used for estimating vibrational modes of the structure. Place the accelerometers at strategic locations to measure the different vibrational modes. During each experiment, the appropriate actuation should be implemented (i.e., sinusoids, white noise, impulse, chirp). For planning of your work, you should follow the following items:

- *First week*: Study laboratory handout. Get familiar with the flexible system, introduction to spectrum analyzer used for experiments. Use **sinusoidal excitation experiments** to estimate location of resonance modes and possible "anti-resonance modes" and their corresponding mode shapes. Estimate and error analysis of resonance frequencies, amplitude ratio and phase shift of accelerometer signals at resonance frequencies.

- *Second week*: Use **chirp or white noise excitation experiments** to estimate the frequency response of flexible structure between the different floors ($H_{21}(j\omega)$: floor 1 to 2 and $H_{31}(j\omega)$: floor 1 to 3) with the spectrum analyzer. Compare the frequencies of the resonance and anti-resonance modes to the results measured in week one. Estimate the parameters of the transfer function models $H_{21}(s)$ and $H_{31}(s)$ (see Section 2.1 – 2.4 of this handout) that accurately models the measured frequency responses between the different floors (floor 1 to 2 and floor 1 to 3). Using Matlab, generate figures that compare the frequency response of your transfer function models with the measured frequency responses. Estimation and error (statistical) analysis of parameters (see Section 2.5 of this handout) in the models.

- *Third week*: Use **chirp or white noise excitation experiments** to estimate the frequency response of the helicopter rotor blade. Using the results from the given FEA model (see Vibration Experiment Lecture), decide upon the best location for the accelerometers to estimate the first and second vibrational modes of the helicopter rotor blade. Estimate the location of resonance modes with spectral analysis and compare to the FEA model.

2 Background information on vibration modeling

2.1 Generalized coordinates and Lagrange's method

Consider the three-story structure depicted in Figure 1. Since each floor has an independent displacement, the generalized coordinates for this 3DOF system are simply the three (independent) positions q_i of the masses $m_i,\ i=1,2,3$.

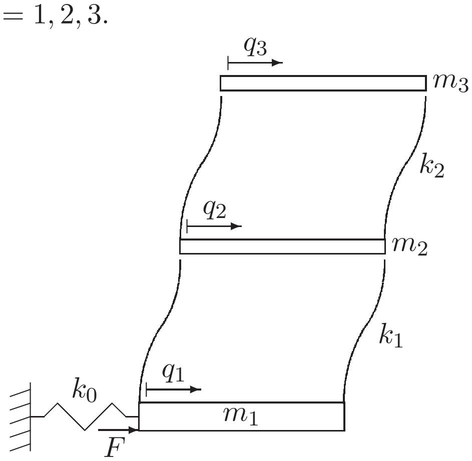

Figure 1: Schematics of simple spring loaded two-story building subjected to base force excitation

Using Lagrange's method[1], the equations of motion for this 3 degree of freedom system (3DOF) can be derived as follows. First of all, the kinetic energy T can directly be expressed in the generalized velocities $\dot{q}_j$:

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2 \tag{1}$$

and it can be seen that T depends only on $\dot{q}$ and not on q. For the formulation of the potential energy we have to rely on the potential energy of a linear shear element of the wall of the structure that is given by $\frac{1}{2}ku^2$, where k is the stiffness of the spring and u is the deformation of the wall. Using this information we directly see that U satisfies

$$U = \frac{1}{2}k_0q_1^2 + \frac{1}{2}k_1(q_2-q_1)^2 + \frac{1}{2}k_2(q_3-q_2)^2 \tag{2}$$

Due to the external force $F(t)$, the generalized forces $Q_i,\ i=1,2$ have to be computed. In equilibrium we see that the total virtual work is given by

$$\delta W = F(t)\delta q_1 \ \Rightarrow\ Q_1 = F(t),\ Q_2 = 0,\ Q_3 = 0$$

The above information can be used for formulate the three scalar Lagrange's equations for $i=1,2,3$. The three Lagrange equations can be combined in a matrix representation

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} k_0+k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} F(t) \tag{3}$$

that describes a (simplified) model of the dynamic behavior of the flexible two-story building.

[1] See also the additional vibration experiment handout on Lagrange's method

2.2 Finite Element Analysis Modeling

The model of three-story building is a typical example of a lumped parameter system with discrete mass, stiffness (and damping) components. Instead of using a lumped parameter system, the displacement $u(x,t)$ of a (continuous) flexible structure is a function of time t and space variable x. To compute the vibration frequencies and vibration modes, Finite Element Analysis (FEA) can be used. FEA is a computer assisted modeling technique based on a finite element method (FEM) in which a structural system is approximated by a finite set of N appropriate elements. The numerical technique of FEM leads to a so-called finite element model in which the displacement $u(x,t)$ (solution to a partial differential equation) is written as a linear combination of admissible functions

$$u(x,t) = \sum_{j=1}^{N} v_j(x) q_j(t)$$

determined by the properties and interpolation of the elements of the finite element model. Describing the displacement field $u(x,t)$ as a linear combination of the product of admissible functions that only depend on space $v_j(x)$ and time $q_j(t)$ allows a finite element model to be written as

$$M\ddot{q}(t) + Kq(t) = QF(t) \tag{4}$$

where M indicates the mass matrix, K the stiffness matrix and $F(t)$ the external forces applied to the boundary of the structure via the input matrix Q.

2.3 Resonance frequencies and mode shapes

It is clear from (3) or a FEA model described in (4) that a vibration model can be decsribed by a set of *coupled second order differential equations* of the form

$$M\ddot{q}(t) + Kq(t) = Qu(t),\ M = M^T > 0,\ K = K^T \geq 0 \tag{5}$$

where M is a symmetric positive definite generalized mass matrix, K is a symmetric positive generalized stiffness matrix and Q is the generalized input matrix due to an external input force $u(t)$. For this special class of dynamic systems it is possible to also write down a set of *decoupled second order differential equations* by means of a coordinate transformation

$$q(t) := Pp(t) \tag{6}$$

of the generalized coordinates $q(t)$. The existence if a matrix P in the coordinate transformation that is able to decouple the Langrange's equations is based on the Linear Algebra result that for any two real symmetric matrices M and K with $M > 0$, there always exists a non-singular matrix P such that

$$P^T MP = I,\ P^T KP = \Omega^2 = \text{diagonal matrix}$$

Post-multiplication of (5) with P^T and substitution of the coordinate transformation (6) in (5) leads to

$$P^T[MP\ddot{p}(t) + KPp(t) = Qu(t)] \ \Rightarrow\ \ddot{p}(t) + \Omega^2 p(t) = \bar{Q}u(t) \tag{7}$$

which is a set of *decoupled second order differential equations.*

Since Ω^2 is a diagonal matrix, we have a set of decoupled second order differential equations

$$\ddot{p}_i(t) + \omega_i^2 p_i(t) = \bar{q}_i u(t)$$

for which the homogeneous solution ($u(t) = 0$) is given by

$$p_i(t) = \sin(\omega_i t)$$

As a result, the diagonal elements ω_i of Ω contain the *resonance frequencies* of the mechanical or flexible structural system.

Since (7) is a set of decoupled equations, an initial condition on the jth element of $p(0)$:

$$\dot{p}(0) = 0, \quad p(0) = \begin{bmatrix} p_1(0) \\ \vdots \\ p_n(0) \end{bmatrix} \text{ with } p_i(0) = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

will lead to dynamic response $p(t)$ in which *only* the jth element of $p(t)$ is non-zero. As a result, an initial condition on the jth element of $p(0)$ the system remains in the *same direction* when considering the response $p(t)$ as an n-dimensional time dependent vector. Since the direction is maintained we have $p(t) = yp(0)$ at any given time, making

$$q_j = Pp(0) = j\text{th column in P}$$

an eigenmode of the structure. Hence, all eigenmodes are given by the columns of the coordinate transformation P in (6). Using this approach, several mode shapes of the helicopter rotor blade can be computed and the three main mode shapes are depicted below.

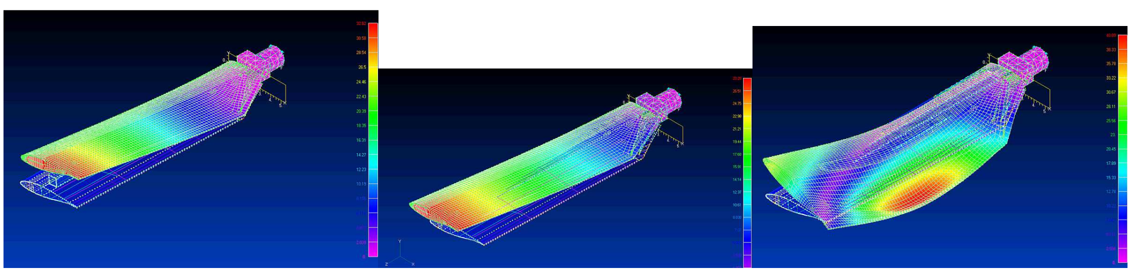

Figure 2: Helicopter blade mode shapes: (left) 1st mode: out-of-plane bending, (middle) 2nd mode: in-plane bending, (right) 3rd mode: torsion

2.4 Transfer function representations

Convert the second-order differential equation (5) to a transfer function representation can be done via a Laplace transform. This allows a characterization of the transfer function $H(s)$ between various generalized coordinates $q_i(s),\ i = 1, 2, \ldots, n$.

Application of the Laplace transform to (5) yields

$$[Ms^2 + K]q(s) = Qu(s)$$

which leads to the transfer function

$$G(s) := [Ms^2 + K]^{-1}Q, \quad q(s) = G(s)u(s)$$

that relates the external input forces to the generalized coordinates. When only the jth generalized coordinate is of importance, $G_j(s)$ can be computed via selection of the jth row in $G(s)$. In case $G_i(s)$ and $G_j(s)$ are scalar, the transfer function between to generalized coordinates is defined as

$$q_j(s) = \frac{G_j(s)}{G_i(s)} q_i(s) := H_{ji}(s) q_i(s)$$

With

$$q_j(s) = G_j(s) u(s)$$

any value for $z \in \mathbb{C}$ for which $G_j(z) = 0$ is a zero of $G_j(s)$. In case z satisfies $z \pm j\omega_z$ (for a system without damping), the system has a "anti-resonance mode" at ω_z and will block a sinusoidal input $u(t) = \sin \omega_z t$ at the generalized coordinate $q_j(t)$. Similar argumentation also holds for

$$q_j(s) = H_{ji}(s) q_i(s)$$

when when examining the transfer function $H_{ji}(s)$. The computation of $G(s)$ for the three-sory flexible structure would require the analytic computation of the inverse of a 3×3 matrix $[Ms^2 + K]$. In case only specific transfer function such as $H_{12}(s)$ from $q_1(s)$ to $q_2(s)$ needs to be computed, direct computation of $H_{12}(s)$ from the Laplace transformation of (3) is more straightforward.

2.5 Resonance and anti-resonance mode frequencies

To understand the location of the resonance modes and the "anti-resonance modes" of the mechanical or flexible structural system one can convert the second-order differential equation to a transfer function representation via a Laplace transform. This will allow you to study the dynamic transfer function $H(s)$ between various generalized coordinates $q_i(s),\ i = 1, 2, \ldots, n$.

Application of the Laplace transform to (3) yields

$$[Ms^2 + K]q(s) = Qu(s)$$

which leads to the transfer function $G(s) = [G_1(s)\ G_2(s)\ G_3(s)]^T$ given by

$$G(s) := [Ms^2 + K]^{-1} Q, \quad q(s) = G(s)u(s)$$

With $Q = [1\ 0\ 0]^T$ as given in (3) it suffices to only compute the first column of the inverse of $[Ms^2 + K]$ and the determinant of $[Ms^2 + K]$ to compute all three transfer functions:

$$\begin{gathered}\mathrm{den}(s) := \det\{Ms^2 + K\} = (m_1 m_2 m_3)s^6 + (k_0 m_2 m_3 + k_1 m_3(m_1 + m_2) + k_2 m_1(m_2 + m_3))s^4 + \\ (k_0 k_1 m_3 + k_0 k_2(m_2 + m_3) + k_1 k_2(m_1 + m_2 + m_3))s^2 + k_0 k_1 k_2\end{gathered}$$

making

$$G(s) = \frac{1}{\mathrm{den}(s)} \cdot \begin{bmatrix} m_2 m_3 s^4 + (k_2 m_2 + (k_1 + k_2)m_3)s^2 + k_1 k_2 \\ k_1(m_3 s^2 + k_2) \\ k_1 k_2 \end{bmatrix}$$

As an example, the transfer function from first floor acceleration $\ddot{q}_1$ to second floor acceleration $\ddot{q}_2$ will be given by

$$q_2(s) = H_{21}(s) q_1(s),\ H_{21}(s) := \frac{G_2(s)}{G_1(s)} = \frac{k_1(m_3 s^2 + k_2)}{m_2 m_3 s^4 + (k_2 m_2 + (k_1 + k_2)m_3)s^2 + k_1 k_2} \tag{8}$$

in which the values of m_1 and k_0 are irrelevant.

To understand the location of the resonance and anti-resonance frequencies, assume $k = k_1 = k_2$ and $m = m_2 = m_3$ for analysis purposes. In that case, $H_{21}(s)$ in (8) reduces to

$$H_{21}(s) = k\frac{ms^2 + k}{m^2s^4 + 3kms^2 + k^2}$$

with the definition of

$$\omega := \sqrt{\frac{k}{m}} \tag{9}$$

we immediately see that $H_{21}(s)$ has a zeros at

$$s_{1,2} = \pm j\omega$$

causing an *"anti-resonance" mode at the frequency of ω rad/s* for the structure when observing $\ddot{q}_2(t)$. To compute the resonance frequencies, we need to compute the roots of the denominator polynomial $m^2s^4 + 3kms^2 + k^2$. With $z := s^2$ this is a standard root finding problem for a second order polynomial

$$m^2z^2 + 3kmz + k^2 = 0$$

for which the roots are given by

$$z_{1,2} = \frac{-3km \pm \sqrt{5}km}{2m^2}$$

Since $\sqrt{5} < 3$ we see that $z_{1,2} < 0$ and with $z = s^2$ we find the roots of $m^2s^4 + 3kms^2 + k^2$ to be

$$s_{1,2} = \pm j\omega\sqrt{\frac{3-\sqrt{5}}{2}},\;\; s_{3,4} = \pm j\omega\sqrt{\frac{3+\sqrt{5}}{2}}$$

where ω was previously defined in (9). As a result, we will observe *two resonance modes at the frequencies* of

$$\omega\sqrt{\frac{3-\sqrt{5}}{2}} \approx 0.618\omega \text{ rad/s and } \omega\sqrt{\frac{3+\sqrt{5}}{2}} \approx 1.618\omega \text{ rad/s} \tag{10}$$

Consequently, the frequencies of the resonance modes will be slightly smaller and larger than the anti-resonance mode frequency observed before. It should be noted that the ratios of 0.168 and 1.618 of the resonance frequencies with respect to the anti-resonance frequency only hold under the assumption that $k_1 = k_2$ and $m_2 = m_3$ and should be verified from the laboratory experiments.

3 Laboratory Report

Below is a list of items that should be addressed in your laboratory work and report. Some of the items and questions refer to the background information on the vibration modeling of the three-story structure mentioned in Section 2.

- Understanding of the resonance modes and mode shapes of the structure using a dynamical model (based on Lagrange's method). What is the lowest resonance frequencies and accompanying model shape in case $k_0 = 0$?
- Estimation of resonance frequencies with statistical (error) analysis using sinusoidal excitation experiments.

- Estimation of amplitude ratio and phase shift of accelerometer signals with excitation signals that have the same frequency as the resonance frequency of the structure. Explain why this information is useful in estimating the mode shapes of the structure.
- Development of a dynamical models that relate acceleration signals $\ddot{q}_i$ of different floors in the structure.
- Frequency response analysis of the flexible structure via spectral analysis. Spectral analysis is done by the HP spectrum analyzer and yields an estimated frequency response. Indicate in the frequency response the various resonance modes found earlier by sinusoidal excitation experiments.
- Frequency response analysis of helicopter rotor blade with spectral analysis. Indicate the type of actuation chosen and where the accelerometers were placed. Explain how the improper placement of an accelerometer could cause a vibrational mode to be unseen in the measurements.

— end of laboratory handout —