Gyroscope Control Experiment – MAE175a

January 11, 2014

Prof. R.A. de Callafon, Dept. of MAE, UCSD

TAs: Jeff Narkis, **email**: jnarkis@ucsd.edu
Gil Collins, **email**: gwcollin@ucsd.edu

Contents

1 Aim and Procedure of Experiment

1.1 Aim of experiment

The moment gyroscope system is a (rigid) mechanical system that consists of a rotating masses mounted in several gimbals. Gyroscopes are used for angular position control of (small) satellites by applying torques via direct-acting, reaction or gyroscopic drive mechanizations within the gyroscope. Each gimbal creates a single Degree Of Freedom (DOF) of an angular position and the moment gyroscope in the laboratory can be configured such that 2 simultaneous input torques can be applied to alter 4 different angular output measurements. In addition, the gyroscope in the laboratory allows for certain DOF's to be constrained via (electromechanical) brakes to simplify the dynamic analysis and control of the gyroscope.

> **The aim of this control experiment is to model and control the 2 angular positions of the inner and outer gymbal of the moment gyroscope by applying a single torque on the rotor drum.**

The *modeling* is used to predict and model the (linearized) dynamic behavior of the gyroscope. The *control* is used to accurately position two different angular positions over a small range of the gyroscope. More background information on the dynamic modeling is given in Section 2. Laboratory experiments to estimate model parameters and validate the model are given in Section 3. Background information on the control design is given in Section 4. A summary on the contents of the laboratory report is given in Section 5.

1.2 Laboratory procedure

The experiment needs to be completed within 3 laboratory session of 3 hours each. During the experiment you are asked to formulate and validate a dynamic models of the gyroscope using a specific Single motor Input and a Single encoder Output (SISO) and develop a linear SISO control algorithm that allows accurate positioning of a specific angular position. Since the dynamic behavior of the gyroscope is inherently non-linear, only small variations of the angular position are considered, but you are also asked to investigate the sensitivity/robustness of your controller for larger angular variations. While performing your experiments during lab hours, it is recommended to follow the order of the tasks listed below.

- first week
  - Introduction to `ECP` software used for experiments and controller implementation.
  - Read this handout and get familiar with the background theory and lab experiments.
  - Consider the following SISO gyroscope configuration. Put Gyroscope in perpendicular position. On control box, turn electromechanical break axis 3 OFF, electromechanical break axis 4 ON. In ECP, turn virtual break axis 2 ON.
  - DO NOT initialize any rotor speed in `ECP`. With rotor motor 1 as input and encoder 3 as output (use `open_loop_motor1.alg` template) propose and perform *open-loop* experiments several times (min. 5) to estimate the unknown parameter(s) of the SISO transfer function $G_{31}(s)$ in (6).

- Think about using encoder 3 velocity as an output for model validation. For the parameter estimation and model validation, keep in mind that $G_{31}(s)$ (with velocity output) can be written as a standard first order system as in (7).
- Validate the model $G_{31}(s)$ via comparison of a measured and a simulated open-loop step response using `maelab` script file.
- With a validated model $G_{31}(s)$, design a PD or PID-controller (with `maelab` script file) satisfying the design requirements of no steady-state error and less than 25% overshoot.
- With `ECP`: implement the controller using `template_axis3_control.alg` and perform *closed-loop* experiments several times (min. 5) to estimate overshoot and settling time of your designed controller.

- week 2

  - Consider now the following SISO gyroscope configuration. Put Gyroscope in perpendicular position. On control box, turn electromechanical break axis 3 ON, electromechanical break axis 4 OFF. In ECP, turn virtual break axis 2 OFF and initialize rotor speed to the speed specified in your `config.txt`
  - With rotor motor 2 as input and encoder 2 as output (use now the template `open_loop_motor2.alg`) propose and perform *open-loop* experiments several times (min. 5) to estimate the unknown parameters of the SISO transfer function $G_{22}(s)$ in (9). For the parameter estimation and model validation, keep in mind that $G_{22}(s)$ can be written as a standard second order system as in (16).
  - Validate the model $G_{22}(s)$ via comparison of a measured and a simulated open-loop step response using `maelab` script file.
  - With a validated model $G_{22}(s)$, design a PD or PID-controller (with `maelab` script file) satisfying the design requirements of no steady-state error and less than 25% overshoot.
  - With `ECP`: implement the controller using `template_axis2_control.alg` and perform *closed-loop* experiments several times (min. 5) to estimate overshoot and settling time of your designed controller.
  - Report how much the nutation frequency (resonant frequency) of the moment gyroscope has been improved in terms of increase of frequency and damping ratio.

- week 3

  - Consider again the following SISO gyroscope configuration. Put Gyroscope in perpendicular position. On control box, turn electromechanical break axis 3 ON, electromechanical break axis 4 OFF. In ECP, turn virtual break axis 2 OFF and initialize rotor speed to the speed specified in your `config.txt`
  - With rotor motor 2 as input and encoder 4 as output (again use the template `open_loop_motor2.alg`) propose and perform *open-loop* experiments several times (min. 5) to estimate the unknown parameters of the SISO transfer function $G_{42}(s)$ in (9).

– Think about using encoder 4 velocity as an output for model validation. For the parameter estimation and model validation, keep in mind that $G_{42}(s)$ (with velocity output) can be written as a standard second order system as in (16).

– Validate the model $G_{42}(s)$ via comparison of a measured and a simulated open-loop step response using `maelab` script file.

– With a validated model $G_{42}(s)$, design a PD or PID-controller (with `maelab` script file) satisfying the design requirements of no steady-state error and less than 25% overshoot.

– With `ECP`: implement the controller using `template_axis4_control.alg` and perform *closed-loop* experiments several times (min. 5) to estimate overshoot and settling time of your designed controller.

– Report how much the nutation frequency (resonant frequency) of the moment gyroscope has been improved in terms of increase of frequency and damping ratio. Report the settling time of the recession of the gyrocope (movement around axis 4).

2 Background Theory on Modeling

2.1 Modeling of the gyroscope

2.1.1 Definition of variables

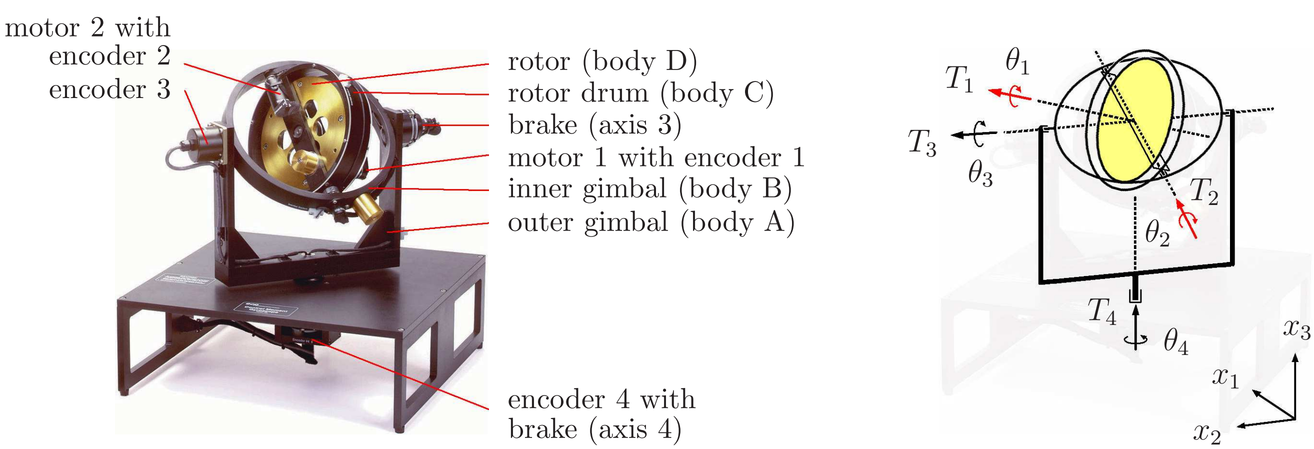

Figure 1: Picture of moment gyroscope (left) and schematics with variables (right).

The gyroscope depicted in Figure 1 consist of 4 (rigid) rotating masses. The 4 rigid bodies each have a angular position θ relative to their rotating gimbal axis and an overview of the definition of the rigid bodies has also been given in Table 2.1.1.

For the derivation of a dynamic model we assume the gyroscope to be symmetric and the center of the rigid bodies to all lie at the center of body D (the rotor). As a result, only the rotational dynamics needs to be taken into account. For the rotational or angular position $\theta_i, i = 1, 2, 3, 4$ of each rigid body, we adopt the following convention.

body	definition	angular position	inertia
A	outer gimbal	θ_4	$\mathbf{I}^A$
B	inner gimbal	θ_3	$\mathbf{I}^B$
C	rotor drum	θ_2	$\mathbf{I}^C$
D	rotor	θ_1	$\mathbf{I}^D$

Table 1: Overview of rigid body elements in gyroscope

2.1.2 Conventions for angular position

- The angular position θ_1 of the rotor (body D) is not of importance. We will only be considering the angular velocity $\omega_1 = \dot{\theta}_1$.
- The angular position θ_2 of the rotor drum (body C) is set to $\theta_2 = 0$ if *the rotor drum (body C) is perpendicular to the inner gimbal (body B).*
- The angular position θ_3 of the inner gimbal (body B) is set to $\theta_3 = 0$ if *the inner gimbal (body B) is perpendicular to the outer gimbal (body A).*
- Since the outer gimbal (body A) is able to rotate freely and the gyroscope is assumed to be symmetric, θ_4 can be reset to $\theta_4 = 0$ at *any angular position of the outer gimbal (body A).*

Since each rigid body might be able to rotate along a 3 dimensional axis, we must consider the inertia I, J and K of each rigid body respectively along x_1, x_2 or x_3 axis. This defines the inertia $\mathbf{I}^b$, with $b = A, B, C$ or D in Table 2.1.1 as

$$\mathbf{I}^b = \begin{bmatrix} I_b & 0 & 0 \\ 0 & J_b & 0 \\ 0 & 0 & K_b \end{bmatrix}, \; b = \text{body } A, B, C \text{ or } D$$

where I_b denotes the inertia along the x_1-axis, J_b denotes the inertia along the x_2 axis and K_b denotes the inertia along the x_3 axis for $b = A, B, C, D$. Note that only moments of inertia are considered, while products on inertia are considered to be zero due to the symmetric nature of the gyroscope.

The angular position of the 4 rigid bodies in the gyroscope can be changed by 2 internal torques and labeled T_1 and T_2 in Figure 1. The 2 internal torques T_1 and T_2 are generated by small DC motors that apply a torque to respectively to the rotor (body D) and the rotor drum (body C). Torque T_1 will make the rotor spin like a wheel around its (perpendicular) axis 1, whereas torque T_2 will make the rotor drum spin around the (longitudinal) axis 2.

2.1.3 Non-linear dynamics of gyroscope

Since both T_1 and T_2 are applied *internally* on a rigid body of the gyroscope, each torque will have a counteracting torque on another rigid body. By inspecting the mechanical connections in the schematics of the gyroscope in Figure 1, the following simple observations can be made:

- Depending on the angular position θ_2 and θ_3, application of the torque T_1 on the rotor (body D) will for example result in a direct counteracting torque T_3 causing rotation of the inner gimbal (body B) and/or a direct counteracting torque T_4 causing rotation of the outer gimbal (body A).
- The angular position of θ_2 can be changed by application of T_2 on the rotor drum (body C). Depending on the angular position θ_3, application of the torque T_2 on the rotor drum (body C) will for example result in a direct counteracting torque T_4 causing rotation of the outer gimbal (body A). Interestingly, when $\omega_1 = \dot{\theta}_1 \neq 0$, rotation of the outer gimbal (body A) is possible even if $\theta_3 = 0$ due to an (indirect) moment caused by a Coriolis force (change in angular momentum).

On the basis of these simple observations, it is clear that the relationship between the internal torques T_1, T_2 and the angular positions θ_i and velocities $\omega_i = \dot{\theta}_i,\ i = 1, 2, 3, 4$ will formulate the equations of motion of the gyroscope. The equations of motions can be derived using Lagrange's equations or Kane's method and will result in a set of coupled (non-linear) differential equations of the form

$$\begin{array}{rcl} T_1 & = & f_1(\theta_2, \theta_3, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) \\ T_2 & = & f_2(\theta_2, \theta_3, \omega_1, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2) \\ 0 & = & f_3(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_3, \dot{\omega}_4) \\ 0 & = & f_4(\theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \omega_4, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \dot{\omega}_4) \end{array} \tag{1}$$

in which the torques T_1 and T_2 are considered as input signals. The full derivation of the (non-linear) equations of motion can be found in the Model 750 Control Moment Gyroscope Manual. It can be noted here that the equations of motion do not depend on θ_1 and θ_4, as the angular position θ_1 of the rotor (body D) and the angular position θ_4 of the outer gimbal (body A) is irrelevant for the dynamic behavior of the gyroscope.

2.1.4 Linearized dynamics of gyroscope

Considering only (small) perturbations around the angular velocity $\omega_1 = \dot{\theta}_1$ of the rotor (body D), the angular position θ_2 of the rotor drum (body C) and the angular position θ_3 of the inner gimbal (body B) allows for a significant simplification of the (non-linear) equations of motion. In case we assume an operating point of the gyroscope with

$$\begin{array}{rcl} \omega_1 & = & \Omega \\ \theta_2 & = & \bar{\theta}_2 \\ \theta_3 & = & \bar{\theta}_3 \end{array}$$

the equations of motion in (1) reduce to

$$
\begin{aligned}
J_D\dot{\omega}_1 &= T_1 - J_D\cos\bar{\theta}_2\dot{\omega}_3 - J_D\sin\bar{\theta}_2\cos\bar{\theta}_3\dot{\omega}_4 \\
(I_C+I_D)\dot{\omega}_2 &= T_2 - J_D\Omega\sin\bar{\theta}_2\omega_3 + J_D\Omega\cos\bar{\theta}_2\cos\bar{\theta}_3\omega_4 + (I_C+I_D)\sin\bar{\theta}_3\dot{\omega}_4
\end{aligned}
$$

$$
\begin{gathered}
(J_B+J_C+J_D-(J_C+J_D-I_D-K_C)\sin^2\bar{\theta}_2)\dot{\omega}_3 = \\
-J_D\cos\bar{\theta}_2\dot{\omega}_1 + J_D\Omega\sin\bar{\theta}_2\omega_2 - J_D\Omega\sin\bar{\theta}_2\sin\bar{\theta}_3\omega_4 - \sin\bar{\theta}_2\cos\bar{\theta}_2\cos\bar{\theta}_3 \\
\\
(I_D+K_A+K_B+K_C+(J_C+J_D-I_D-K_C)\sin^2\bar{\theta}_2 + \\
(I_B+I_C-K_B-K_C-(J_C+J_D-I_D-K_C)\sin^2\bar{\theta}_2)\sin^2\bar{\theta}_3)\dot{\omega}_4 = \\
-J_D\sin\bar{\theta}_2\cos\bar{\theta}_3\dot{\omega}_1 - J_D\Omega\cos\bar{\theta}_2\cos\bar{\theta}_3\omega_2 + (I_C+I_D)\sin\bar{\theta}_3\dot{\omega}_2 + \\
J_D\Omega\sin\bar{\theta}_2\sin\bar{\theta}_3\omega_3 - (J_C+J_D-I_D-K_C)\sin\bar{\theta}_2\cos\bar{\theta}_2\cos\bar{\theta}_3\dot{\omega}_3
\end{gathered}
\tag{2}
$$

Although the equations in (2) look complicated, they have all been written in the form where the inertia times angular acceleration equals the sum of torques:

$$I\ddot{\theta}_i = \sum T$$

reflecting 2nd Newton's law for rotational motion. The coupled set of (linear) differential equations are useful in determining the linear dynamic model of the gyroscope for special cases.

2.1.5 Linear dynamics for special case in laboratory experiment

To further simplify the dynamical model of the gyroscope, we consider several special cases on the basis of an operating point of the gyroscope given by

$$
\begin{aligned}
\omega_1 &= \Omega \\
\theta_2 &= \bar{\theta}_2 = 0 \\
\theta_3 &= \bar{\theta}_3 = 0
\end{aligned}
\tag{3}
$$

where zero angles are defined according to the convention defined on page 5 of this laboratory handout. With the operating point defined in (3), special cases of the dynamics of the gyroscope are found by applying some of the (electromechanical) brakes for either axis 3 (rotation of outer gimbal) or axis 4 (rotation of inner gimbal) of the gyroscope.

In case none of the brakes are used, both the inner and outer gimbals are able to rotate freely. With the operating point defined in (3), the linearized equations of motion in (2) reduce to

$$
\begin{aligned}
J_D\dot{\omega}_1 &= T_1 - J_D\dot{\omega}_3 \\
(I_C+I_D)\dot{\omega}_2 &= T_2 + J_D\Omega\omega_4 \\
(J_B+J_C+J_D)\dot{\omega}_3 &= -J_D\dot{\omega}_1 \\
(I_D+K_A+K_B+K_C)\dot{\omega}_4 &= -J_D\Omega\omega_2
\end{aligned}
\tag{4}
$$

and formulate a set of couple linear (second order) differential equations. This can also be written as a *set of coupled first order differential equations* by definition of the state vector:

$$
\begin{gathered}
x(t) = \begin{bmatrix} \theta_2(t) & \theta_3(t) & \theta_4(t) & \omega_1(t) & \omega_2(t) & \omega_3(t) & \omega_4(t) \end{bmatrix}^T \\
\text{and the input}: \; u(t) = \begin{bmatrix} T_1(t) & T_2(t) \end{bmatrix}^T
\end{gathered}
$$

formulating the state space model

$$\dot{x}(t) = Ax(t) + Bu(t), \text{ with}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{J_D\Omega}{I_C+I_D} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-J_D\Omega}{I_D+K_A+K_B+K_C} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{J_B+J_C+J_D}{J_D(J_B+J_C)} & 0 \\ 0 & \frac{1}{I_C+I_D} \\ \frac{-1}{J_B+J_C} & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

The result is a 7th order state space model. Computing the eigenvalues of the state matrix A results in the following pole locations:

- 2 poles at 0 due to the rigid body mode. The rigid body mode is due to free rotation θ_3 of the inner gimbal (body B) as a result of a direct counteracting torque T_3 caused by torque T_1 to rotate the rotor (body D).
- 3 additional poles at 0 due to the kinematic differential equations.
- 2 complex poles that models the oscillatory behaviour that couples the rotor drum (body C) rotation with the rotation of the outer gimbal (body A). The resonance frequency ω_n found from the complex pole pair located at $\pm j\omega_n$ is called the *nutation frequency of the gyroscope* in rad/s.

Instead of writing a state space model (5), the linearized equations (4) can also be used to write a transfer function representation between the angular positions and applied torques via Laplace transform. Some of the resulting transfer function are

$$\begin{aligned}
\theta_2(s) &= G_{22}(s)T_2(s), \; G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} \\
\theta_3(s) &= G_{31}(s)T_1(s), \; G_{31}(s) = -\frac{1}{(J_B + J_C)s^2} = -\frac{K}{s^2} \\
\theta_4(s) &= G_{42}(s)T_2(s), \; G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s}
\end{aligned}$$

The transfer function $G_{31}(s)$ is the relation between $\theta_3(s)$ of the inner gymbal (body B) and the applied torque $T_1(s)$ on the rotor (body D) and indicates a simple rigid body motion (2 poles at origin) due to free rotation θ_3 of the inner gimbal (body B). Transfer function $G_{22}(s)$ is the relationship between torque T_2 and the resulting angular position θ_2 of the rotor drum (body C).

2.2 Gyroscope dynamics for weekly lab experiment

Week 1: application of axis 4 brake

During week 1 of the experiments, you will be engaging axis 4 brake to eliminate the rotational freedom θ_4 of the outer gymbal (body A). By setting $\omega_4 = \dot{\omega}_4 = 0$, the linearized equations in (4) for the operating point (3) reduce to

$$\begin{aligned}
J_D\dot{\omega}_1 &= T_1 - J_D\dot{\omega}_3 \\
(I_C + I_D)\dot{\omega}_2 &= T_2 \\
(J_B + J_C + J_D)\dot{\omega}_3 &= -J_D\dot{\omega}_1
\end{aligned}$$

The equations can be rewritten into transfer function format

$$\begin{aligned}
\theta_1(s) &= G_{11}(s)T_1(s),\ G_{11}(s) = \frac{J_B + J_C + J_D}{J_D(J_B + J_C)s^2} \\
\theta_2(s) &= G_{22}(s)T_2(s),\ G_{22}(s) = \frac{1}{(I_C + I_D)s^2} \\
\theta_3(s) &= G_{31}(s)T_1(s),\ G_{31}(s) = \frac{-1}{(J_B + J_C)s^2}
\end{aligned} \tag{6}$$

and indicate the lack of a nutation frequency ω_n (11) in the transfer function models. In addition, it can be observed that:

- The transfer functions do not depend on $\omega_1 = \Omega$ and one can set $\omega_1 = 0$ (zero initial rotor speed).
- The equation for ω_2 is completely decoupled from the other two equations involving ω_1 and ω_3. Hence, we can also eliminate the rotational freedom θ_2 of the rotor drum (body C) without influencing the transfer functions $G_{11}(s)$ or $G_{31}(s)$ in (6).

Setting $\omega_1 = 0$ by setting zero initial rotor speed in `ECP` and eliminating the rotational freedom θ_2 of the rotor drum (body C) by enabling the virtual brake in `ECP` will be done during our week 1 experiments. As a result, all transfer functions in (6) can be written in the general form

$$\theta_i(s) = G_{i1}(s)T_1(s), \quad G_{i1}(s) = \frac{K_0}{s^2 + \beta_0 s}, \quad i = 1, 2, 3$$

where a gain K_0 is used to model the gain (amplification) and an extra β_0 (inverse time constant) can be used to model additional damping in the rigid model. Interestingly, if we measure the angular velocity $\omega_i(t) = \frac{d}{dt}\theta_i(t), \quad i = 1, 2, 3$ we see that the transfer functions reduce to

$$\omega_i(s) = s \cdot G_{i1}(s)T_1(s), \quad s \cdot G_{i1}(s) = \frac{K_0}{s + \beta_0}, \quad i = 1, 2, 3 \tag{7}$$

making the relation between the input torque $T_1(t)$ and the angular velocity $\omega_i(t)$ a simple first order system.

Week 2 & 3: application of axis 3 brake

During week 2 and 3 you will be engaging the axis 3 brake to eliminate the rotational freedom θ_3 of the inner gimbal (body B). This configuration is useful in week 2 for examining and demonstrating the nutation frequency, provided $\dot{\theta}_1 = \omega_1 = \Omega > 0$, by observing the angular position θ_2 of the rotor drum (body C). In addition, this configuration is useful in week 3 to study the gyroscopic torque action, provided $\dot{\theta}_1 = \omega_1 = \Omega > 0$. In this case, the angular position θ_4 of the outer gimbal (body A) can be controlled by changing the angular position θ_2 of the rotor drum (body C) with a torgue $T)2$ generated by motor 2.

By setting $\omega_3 = \dot{\omega_3} = 0$, the linearized equations in (4) for the operating point (3) reduce to

$$\begin{aligned} J_D\dot{\omega}_1 &= T_1 \\ (I_C + I_D)\dot{\omega}_2 &= T_2 + J_D\Omega\omega_4 \\ (I_D + K_A + K_B + K_C)\dot{\omega}_4 &= -J_D\Omega\omega_2 \end{aligned} \tag{8}$$

Since the last two equations in (8) have not changed compared to (4), the transfer functions from $T_2(s)$ to $\theta_2(s)$ and $\theta_4(s)$ remain the same. The resulting transfer functions for this case can be written as

$$\begin{aligned} \theta_1(s) &= G_{11}(s)T_1(s),\ G_{11}(s) = \frac{1}{J_D s^2} \\ \theta_2(s) &= G_{22}(s)T_2(s),\ G_{22}(s) = \frac{I_D + K_A + K_B + K_C}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^2 + \Omega^2 J_D^2} \\ \theta_4(s) &= G_{42}(s)T_2(s),\ G_{42}(s) = \frac{-\Omega J_D}{(I_C + I_D)(I_D + K_A + K_B + K_C)s^3 + \Omega^2 J_D^2 s} \end{aligned} \tag{9}$$

It can be observed that the first equation is independent from the other equations. As a result, the rotor speed $\omega_1(s) = s\theta_1(s)$ may be controlled independently via the torque T_1 generated by the rotor motor 1 via a model $s\cdot G_{11}(s)$ that is a simple integrator. We choose to control $\dot{\theta}_1(t) = \omega_1(t)$ to fixed value $\omega_1(t) = \Omega > 0$ in our week 2 & 3 laboratory experiments.

With the fixed rotor speed $\omega_1(t) = \Omega > 0$ the transfer function $G_{22}(s)$ can be written as a standard second order system

$$\theta_2(s) = G_{22}(s)T_2(s), \quad G_{22}(s) = K_1 \cdot \frac{\omega_n^2}{s^2 + 2\beta_1\omega_n s + \omega_n^2} \tag{10}$$

where K_1 denotes a gain (amplification) and an extra β_1 (damping ratio) can be used to model additional damping in the gyroscope. This makes the relation between the input torque $T_2(t)$ and the angular velocity $\omega_2(t)$ a simple second order system. In this second order system we have two (complex conjugate) poles that model the **(undamped) nutation frequency** ω_n of the gyroscope (theoretically) given by

$$\omega_n = \frac{\Omega J_D}{\sqrt{(I_C + I_D)(I_D + K_A + K_B + K_C)}} \tag{11}$$

and depends only on specific body inertia and the (fixed) rotor speed $\omega_1(t) = \Omega > 0$.

Transfer function $G_{42}(s)$, relating the torque $T_2(s)$ and the angular position $\theta_4(s)$ of the outer gimbal (body A), exhibits the same complex pole pair with the nutation frequency ω_n in (11). In addition, there is a pole at 0 (integrator) that causes θ_4 eventually to ramp (up/down) linearly whenever a constant torque T_2 is applied, making

$$G_{42}(s) = K_2 \cdot \frac{\omega_n^2}{s^2 + 2\beta_2\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

where again a K_2 is used to denote a gain (amplification) in the system and an extra β_2 (damping ratio) can be used to model additional damping in the gyroscope. Interestingly, if we measure the angular velocity $\omega_4(t) = \frac{d}{dt}\theta_4(t)$ during our week 3 experiment, we see that the transfer function reduces to

$$\omega_4(s) = s \cdot G_{42}(s)T_2(s), \quad s \cdot G_{42}(s) = K_2 \cdot \frac{\omega_n^2}{s^2 + 2\beta_2\omega_n s + \omega_n^2} \tag{12}$$

making the relation between the input torque $T_2(t)$ and the angular velocity $\omega_4(t)$ again a simple second order system.

2.3 Dynamic response of first order and second order systems

As indicated in the previous section, measuring the angular velocity $\omega_i(t) = \frac{d}{dt}\theta_i(t), \quad i = 1, 2, 3$ during our first week laboratory experiment enables us the describe the gyroscope dynamics with a simple 1st order system given in (7). In week 2, measuring the angular position $\theta_2(s)$ allows us the describe the gyroscope dynamics with a standard 2nd order system given in (10). And finally, measuring the angular velocity $\omega_4(t) = \frac{d}{dt}\theta_4(t)$ during our third week laboratory experiment again allows us the describe the gyroscope dynamics with a standard 2nd order system given in (12).

To estimate the model parameters (week 1: K_0, β_0, week 2: K_1, ω_n, β_1 and week 3: K_2, ω_n, β_2) we will perform simple dynamic experiments in which we measure the **step response** of the gyroscope. In order to understand how to obtain the model parameters from the step response, we first need to write down the actual step response as a function of the (unknown) model parameters. Given the dynamics of $s \cdot G_{i1}(s)$ in the 1st week and $G_{22}(a)$ and $\cdot G_{42}(s)$ in the 2nd and thrid week laboratory experiments, we only need to know the step response of a (standard) first and second order system. The results are summarized below.

First consider a 1st order system with a transfer function

$$y(s) = G(s)u(s), \quad G(s) = \frac{K}{s + \beta}, \quad i = 1, 2, 3$$

then a step input $u(t) = U, \ t \geq 0$ of size U results in the output response

$$y(t) = \frac{K}{\beta} \cdot U \cdot \left[1 - e^{-\beta t}\right] \tag{13}$$

The result follows directly from inverse Laplace transform of

$$y(s) = G(s) \cdot \frac{U}{s} = \frac{KU}{s + \beta} \cdot \frac{1}{s}$$

Secondly, consider a 2nd order system with a transfer function

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2}$$

then a step input $u(t) = U,\ t \geq 0$ of size U on the 2nd order system results in the output response

$$y(t) = K \cdot U \cdot \left[1 - \mathrm{e}^{-\beta\omega_n t}\sin(\omega_d t + \phi)\right] \tag{14}$$

where

$$\begin{aligned} \omega_d &= \omega_n\sqrt{1-\beta^2} && \text{damped resonance frequency in rad/s} \\ \phi &= \tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta} && \text{phase shift of response in rad} \end{aligned}$$

The computation of the step responses in (13) and (14) can be used to determine the unknown model parameter (week 1: K_0, β_0, week 2: K_1, ω_n, β_1 and week 3: K_2, ω_n, β_2) using simple step response experiments during the laboratory.

3 Laboratory Experiments

3.1 Estimation of model parameters

For the completion of the dynamical model of the gyroscope, basically all the moments of inertia ($J_b,\ I_b$ and $K_b,\ b = A, B, C, D$) would have to be determined. However, for some of the special cases discussed in Section 2.1.5, only a subset of all the moments of inertia needs to be determined to find the transfer function models. Moreover, the transfer function models can be reduced down to simple 1st and 2nd order transfer functions as indicated in (7) for week 1, (10) for week 2 and (12) for the week 3 experiments in the lab. As a result, only the lump sum or ratio of certain moments of inertia are relevant to find the coefficients of a particular (first and second order) transfer function of the gyroscope given. These coefficients are the model parameters you would have to estimate during your laboratory experiments and they will be determined with simple step response experiments.

To find the coefficients of a particular (first order) transfer function of the gyroscope, consider a transfer function of a standard 1st order system

$$G(s) = \frac{K}{s+\beta} \tag{15}$$

similar as in (7), where K denotes a gain, β is inverse time constant that models (possible) damping of the rigid body mode of the gyroscope. With this transfer function, a Voltage step input $u(t)$ of size U will results in the analytic solution given in (13).

Using the analytic solution given in (13) and

$$\left.\frac{d}{dt}y(t)\right|_{t=0} = \left.K \cdot U\mathrm{e}^{-\beta t}\right|_{t=0} = K \cdot U$$

along with the typical step response $y(t)$ of a lightly damped velocity response of a rigid body system indicated in Figure 2, it can be seen that the gain K and the (inverse time constant) β can be estimated via

$$\hat{K} = \frac{1}{U} \cdot \left.\frac{d}{dt}y(t)\right|_{t=0}$$

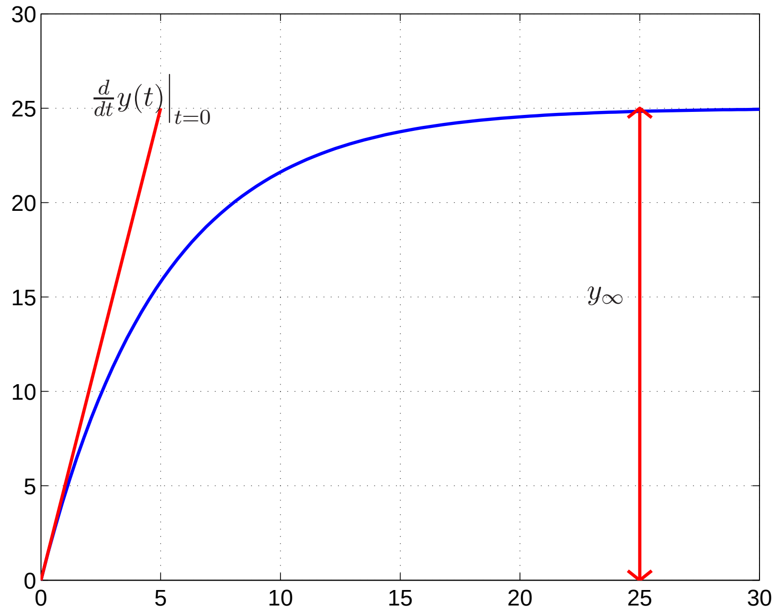

Figure 2: Step response of (velocity) of lightly damped gyroscope system with rigid body motion

and knowing that the (velocity) $\lim_{t\to\infty} y(t) = y_\infty = \frac{K}{\beta} \cdot U$ we find

$$\hat{\beta} = \frac{\hat{K}}{y_\infty} \cdot U$$

To find the coefficients of a particular (second order) transfer function of the gyroscope, consider a transfer function of a standard 2nd order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \tag{16}$$

similar as in (10) and (12) where K denotes a gain, ω_n indicates the (undamped) nutation frequency as in (11) and $0 \leq \beta < 1$ is a damping ratio that models (possible) damping of the nutation mode of the gyroscope. With this transfer function, a Voltage step input $u(t)$ of size U will results in the analytic solution given in (14).

Using the analytic solution given in (14) and the typical step response $y(t)$ of a lightly damped second order system indicated in Figure 3, it can be seen that ω_d and the product of $\beta\omega_n$ can be estimated via

$$\begin{aligned} \hat{\omega}_d &= 2\pi \frac{n}{t_n - t_0} \\ \widehat{\beta\omega_n} &= \frac{1}{t_n - t_0} \ln\left(\frac{y_0 - y_\infty}{y_n - y_\infty}\right) \end{aligned}$$

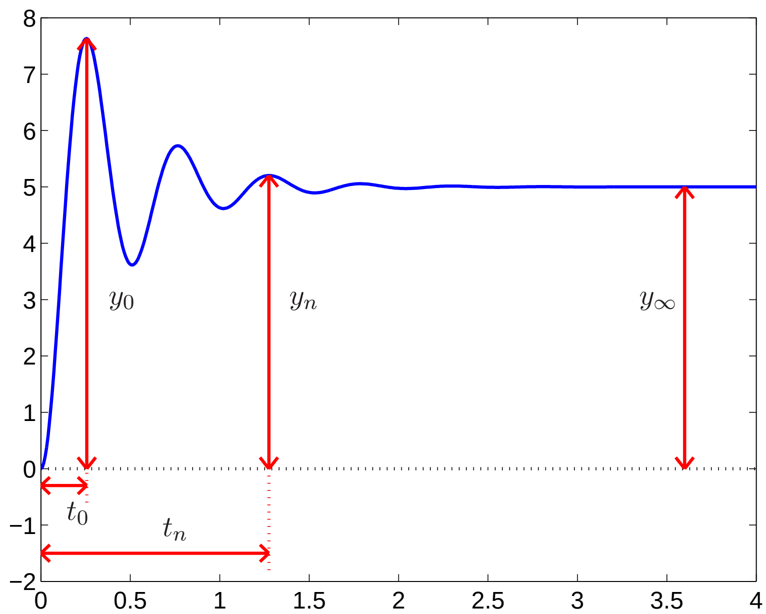

Figure 3: Step response of lightly damped coupled gyroscope system with nutation frequency.

and as a result, ω_n, β and K in (16) can be estimated by

$$\begin{aligned}
\hat{\omega}_n &= \sqrt{\hat{\omega}_d^2 + (\widehat{\beta\omega_n})^2} \\
\hat{\beta} &= \frac{\widehat{\beta\omega_n}}{\hat{\omega}_n} \\
\hat{K} &= \frac{y_\infty}{U}
\end{aligned}$$

As a result, simple step response experiments will allow you to directly estimate the coefficients of the second order transfer function.

When conducting your (step response) experiments in the lab to find the model parameters unknown model parameters (week 1: K_0, β_0, week 2: K_1, ω_n, β_1 and week 3: K_2, ω_n, β_2) keep the following items in mind:

- One has to realize that each of the transfer functions given in Section 2.1.5 relates torque (measured in Nm) to angular position (measured in rad/s). Since we apply torque via Voltage applied to a DC-motor and measure angular position in encoder counts, each transfer function will have a different gain K_0, K_1 and K_2 with the units encoder counts/Voltage.
- Perform (step response) experiments and the estimation of the unknown model parameters several times (at least 5 times) to allow for statistical analysis. Mean value, standard deviation and confidence intervals (based on t-distribution) will have to be part of your report.

3.2 Validation of model

In order to validate the model parameters that you are estimating, one should *compare measured data* with *data simulated by your model*. Validation of the model is crucial before continuing with the design of a controller and can be done by editing the `models.m` file and running the `maelab.m` script file available on the computer in the laboratory.

Keep in mind that you will use a transfer function model (week 1: $G_{i1}(s)$, week 2: $G_{22}(s)$ and week 3: $G_{42}(s)$) to design your controller. With an invalid or invalidated model, the controller cannot be designed reliably. Consider the following items for the validation of your model:

- Propose experiments and the signals to be measured to validate your models that you are constructing.
- You can simulate the step response of your model (week 1: $G_{i1}(s)$, week 2: $G_{22}(s)$ and week 3: $G_{42}(s)$) with the `maelab.m` script file. All you need to do is modify the model parameters (week 1: K_0, β_0, week 2: K_1, ω_n, β_1 and week 3: K_2, ω_n, β_2) in the file `models.m` available on the computer in the laboratory.
- Which signals do you need to save, in order to be able to simulate data with your model and to compare simulated data with measured data? Think about *input* and *output* signals.
- For the week 2 & 3 experiments, ensure that the measured (damped) nutation frequency

  $$\omega_n = \frac{\Omega J_D}{\sqrt{(I_C + I_D)(I_D + K_A + K_B + K_C)}}$$

  of the gyroscope is close to the (damped) nutation frequency of your model.
- Comment on how good the model is able to simulate the measured data. Give suggestions in case you need to adjust the model parameters in your report.

In order to compare a measurement with a simulation, you have to save your measured data of your relevant dynamic experiments in a text file (extension `.txt`) via the `export raw data` option in the `ECP` software. Do not start the filename with a number and make sure you save your data under your working directory! Subsequently you have to prepare the text file of the saved data so that it can be read it into Matlab. This is done by the following editing steps:

1. First line in text file: Comment out the first line with a `%`
2. Second line in text file: Enter `dummy=` before the opening bracket `[`
3. Last line in text file: Put a semicolon `;` behind the closing bracket `]`
4. After last line in text file: define time t, input u and output y by selecting the appropriate columns from the `dummy` variable. For example, if you have selected to save the control effort (input u) and the encoder 1 position (output y), this can be done by adding the following lines to the end of the text file:

```
t=dummy(:,2);
y=dummy(:,a); % select the right column with the variable a
u=dummy(:,b); % select the right column with the variable b
clear dummy;
```

5. Save the raw text file as a file with the extension `.m`

The result is a Matlab script file that can be run from Matlab. When you run the file, the time vector `t`, the input signal `u` and the output signal `y` are available in the Matlab workspace for plotting and comparison purposes to validate your model. The script file `maelab.m` can also directly read the data files for comparison purposes and the editing steps above are only necessary if you like to plot the data individually in Matlab to create figures for your report.

4 Design of Feedback Control for Gyroscope

4.1 Control specification

You are asked to design a control algorithm for the three different (special) cases of the gyroscope system as discussed in Section 2.1.5. In order of complexity, control algorithms should be designed on the basis of a model of the gyroscope for the following situations:

1. **Week 1: Axis 4 brake with feedback of encoder 3 to motor 1**
   Design a PD feedback controller that uses θ_3 (encoder 3) as measurement and T_1 (motor 1) as control effort that positions the inner gymbal (body B) as fast as possible. Show, by making 10 to 20 degree steps on the command signal that θ_3 settles with less than 25% overshoot.

2. **Week 2: Axis 3 brake with feedback of encoder 2 to motor 2**
   Design a P or PD- feedback controller that uses θ_2 (encoder 2) as measurement and T_2 (motor 2) as control effort that positions the inner gymbal (body B) as fast as possible. Show, by making 10 to 20 degree steps on the command signal that θ_2 settles with less than 25% overshoot.

3. **Week 3: Axis 3 brake with feedback of encoder 4 to motor 2**
   Design a feedback controller that uses θ_4 (encoder 4) as measurement and T_2 (motor 2) as control effort that positions the outer gymbal (body A) as fast as possible. Show, by making 10 to 20 degree steps on the command signal that θ_4 settles with less than 25% overshoot.

4.2 Programming the control algorithm

In the `ECP` software of the gyroscope control system, control algorithms are specified by means of simple programs that are compiled and downloaded to the DSP. The control algorithms is made up of three distinct sections:

- Definition segment – used to assign (internal) variable `g1` till `q100` to the user variables defined in the definition segment. Variables can include servo gain and measurements to be saved for control and each line should be formatted according to the example:

  ```
  #define gain_1 q2  ; assigns gain_1 to the (internal) variable q2
  ```

  Next to the 100 internal variables there are 8 global variables that cannot be defined:

  ```
  cmd1_pos        ; the commanded position (reference) for encoder 1
  cmd2_pos        ; the commanded position (reference) for encoder 2
  enc1_pos        ; encoder 1 output
  enc2_pos        ; encoder 2 output
  enc3_pos        ; encoder 3 output
  enc4_pos        ; encoder 4 output
  control_effort1 ; the input to motor 1
  control_effort2 ; the input to motor 2
  ```

- Variable initialization segment – used to assign numerical values to the variables defined in the definition segment. Lines should be formatted according to the example

  ```
  gain_1=0.78     ; assigns the value 0.78 to variable gain_1 = q2
  ```

- Servo loop or real-time execution segment – this segment starts with a `begin` and ends with an `end` statement. All code between the `begin` and `end` statement will be executed *every sample period* for real-time control implementation. A simple PD controller can be implemented with a code similar to

  ```
  begin
       control_effort2=Kp*(cmd_pos-enc2_pos)+Kd*(enc2_pos-past_enc2_pos)
       past_enc2_pos=enc2_pos
  end
  ```

  Note that the code is *case insensitive*.

4.3 Design of controllers

For the design of the P-controller, heuristic tuning rules such as Ziegler and Nichols can be used. Alternatively, with the knowledge of the transfer function model of the gyroscope, a stabilizing controller can be found via the root-locus method or with the frequency domain method. Irrespective of the method you use to design your control algorithm, address the following items in your report:

- Explain the method you use to design your controller and how you evaluate the stability of your closed-loop system. Specifically mention the **amplitude and phase margin** of your control design.

- Give a well motivated choice for the values of k_p and k_d of your PD-controller. Explain the improvements in damping and reduction of overshoot you have obtained by the use of a PD-controller.
- Is the PD-controller able to position θ_2 without any steady state error? Give an explanation of your results in terms of the resulting steady state error and explain why you had to choose a PID controller to get rid of the steady state error.

In case a steady state error is still present in your closed-loop step response, an additional integral action in the controller is able to reduce steady state errors. For that purpose, the PD-controller is extended to a PID-controller. For the design of a PID-controller give a well motivated choice for the values of k_p, k_d and k_i of your PID-controller in your report. Explain the reduction in steady-state error by the use of a PID-controller.

4.4 Sensitivity analysis

Once you have designed a properly working feedback controller, you should check the sensitivity of the feedback system. Since the controller is based on a model that assumes small variations around the operating point (3), you are asked to make perturbations of the operating point (3) and verify the stability and performance of your designed feedback controllers. **Be careful, the feedback system might become unstable and in that case you should turn off the controller as soon as possible.**
Address the following items in your report:

- How sensitive is your designed feedback system to changes in θ_3 and $\omega_1 = \Omega$? Create a table/graph comparing settling time and overshoot as a function of θ_3 and $\omega_1 = \Omega$.
- How sensitive is your designed feedback system to control parameter variations? Make 10% variations in your designed control parameters and observe changes in settling time and overshoot.
- Do you have an explanation for the sensitivity to the various parameters? Comment on the quality requirements of the model.

5 Laboratory Report

By the end of the gyrocope control experiment you should have estimated (with error analysis) and validated the parameters of a dynamic model of the gyroscope for the various cases. You should also have a stabilizing feedback controllers for the three different gyroscope configurations studied during the 3 weeks of experiments.

The results of your laboratory work have to be reported in a coherent written report. The report will be graded on the layout and the contents that should (at least) contain the following items:

- Estimation and error analysis of parameters in the model of the gyroscope system.
- Comparing dynamic experiments with dynamic model simulations.

- Design procedure for PD, PID (or state feedback) controller on the basis of a dynamic model of the system to be controlled
- Implementation of feedback controllers and comparing closed-loop dynamic experiments with closed-loop dynamic model simulations
- Sensitivity (robustness) analysis by (closed-loop) experiments that examine settling time and overshoot variations due to changes in operating point variations

In addition, refer to the questions posed throughout this laboratory handout. Make sure to include your answers to these questions in the appropriate sections of your laboratory report.

— end of laboratory handout —