MAE108 S2014 - Homework 6 Solutions

Problem 3.56

Seismic capacity C is lognormal with median, $x_m = 6.5$ and standard deviation, $\sigma = 1.5$
For lognormal distribution, $\mu > x_m$; Hence

$$\frac{\sigma}{\mu} < \frac{\sigma}{x_m} = \frac{1.5}{6.5} \simeq 0.23 < 0.3 \rightarrow \zeta \simeq \delta = \frac{\sigma}{\mu}$$

Express the lognormal parameter λ

$$\lambda = \ln x_m = \ln 6.5 = 1.87$$

Express the lognormal parameter λ in other way,

$$\lambda = \ln \mu - \frac{1}{2}\left(\frac{1.5}{\mu}\right)^2$$
$$1.87 = \ln \mu - \frac{1}{2}\left(\frac{1.5}{\mu}\right)^2$$

$\mu \simeq 6.7$ by trial and error method or do search (www.wolframalpha.com) the approximated value.
and lognormal parameter,$\zeta = \frac{\sigma}{\lambda} \simeq \frac{1.5}{6.7} = 0.22$

a) Calculate the probability of damage when subjected to maximum earthquake.

$$P(C < 5.5) = \Phi\left(\frac{\ln x - \lambda}{\zeta}\right) = \Phi\left(\frac{\ln 5.5 - \ln 6.5}{0.22}\right) = \Phi\left(\frac{-0.165}{0.22}\right) = \Phi(-0.75) = 0.227$$

b) Calculate the conditional probability that the building would fail when subjected to maximum earthquake provided it withstood a moderate earthquake of force facor 4. Failure occurs when force factor exceeds 5.5.
Calculate the probability of failure

$$\begin{aligned} P(C < 5.5|C > 4) &= \frac{P(C > 4 \cap C < 5.5)}{P(C > 4)} = \frac{P(4 < C < 5.5)}{P(C > 4)} \\ &= \frac{\Phi\left(\frac{\ln 5.5 - \ln 6.5}{0.22}\right) - \Phi\left(\frac{\ln 4.0 - \ln 6.5}{0.22}\right)}{1 - \Phi\left(\frac{\ln 4 - \ln 6.5}{0.22}\right)} \\ &= \frac{\Phi(-0.75) - \Phi(-2.91)}{1 - \Phi(-2.91)} = \frac{0.227 - 0.0019}{1 - 0.0019} \\ &= 0.225 \end{aligned}$$

c) Mean rate of damaging earthquake, ν

$$\begin{aligned} \nu &= \text{probability damage} \times \text{return period} \\ &= 0.2 * \frac{1}{500} \\ &= 0.0004 \end{aligned}$$

Calculate the probability that building survives for 100 years

$$\begin{aligned} P(\text{building survives 100 years}) &= P(\text{no damaging earthearthquake in 100 years}) \\ &= e^{-0.0004*100} \\ &= e^{-0.04} \\ &= 0.96 \end{aligned}$$

d) Calculate the probability that a structure will survive.

$$
\begin{aligned}
&P(\text{survival of at least 4 structures during the earthquake})\\
&= \binom{5}{4}(0.8)^4(0.2)^1 + \binom{5}{5}(0.8)^5(0.2)^0\\
&= 0.4096 + 0.32747 = 0.737
\end{aligned}
$$

The mean rate earthquake that causes damage to at most 3 structures is
$= 0.737 * \frac{1}{500} = 0.00147$
$P(\text{at least four of the five buildings will survive 100 years}) = e^{-0.00147*100} = e^{-0.147} = 0.863$

Problem 3.58

a) The marginal density function of the daily water level of reservior A, X is $f_X(x)$
$f_X(x)$ is obtained by "integrating out" the dependence on y,
Calculate the marginal density function of the daily water level for reservoir A.

$$f_X(x) = \int_0^1 \frac{6}{5}(x+y^2)dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1 = \frac{2}{5}(3x+1) \ \ (0 < x < 1)$$

b) Calculate the probability density function for the conditions

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(6/5)(x+y^2)}{(2/5)(3x+1)} = \frac{3(x+y^2)}{3x+1}$$

$$
\begin{aligned}
\text{Hence } P(Y > 0.5|X = 0.5) &= \int_{0.5}^1 f_{Y|X}(y|x)dy\\
&= 3\int_{0.5}^1 \frac{0.5+y^2}{1.5+1} = \frac{3}{2.5}\left[0.5y + \frac{y^3}{3}\right]_{0.5}^1\\
&= 0.65
\end{aligned}
$$

c) Calculate the joint second moment of X and Y

$$
\begin{aligned}
E(XY) &= \int_0^1\int_0^1 xyf_{X,Y}(x,y)dxdy = \frac{6}{5}\int_0^1\int_0^1 (x^2y + xy^3)dxdy\\
&= \frac{2}{5}\int_0^1 ydy + \frac{3}{5}\int_0^1 y^3dy = 1/5 + 3/20 = 7/20 = 0.35
\end{aligned}
$$

Calculate the joint second central moment.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Here, second central moment is $Cov(X,Y)$

$$
\begin{aligned}
Cov(X,Y) &= E(XY) - \int_0^1 xf_X(x)dx * \int_0^1 yf_Y(y)dx\\
&= 0.35 - \int_0^1 x\frac{2}{5}(3x+1)dx * \int_0^1 y\frac{3}{5}(2y^2+1)dy\\
&= 0.35 - \frac{2}{5}\left[x^3 + \frac{x^2}{2}\right]_0^1 * \frac{3}{5}\left[\frac{y^4}{2} + \frac{1}{2}\right]_0^1\\
&= 0.35 - \frac{2}{5}\left(\frac{3}{2}\right) * \frac{3}{5}(1)\\
&= 0.35 - 0.6 * 0.6\\
&= -0.01
\end{aligned}
$$

Calculate the mean square value of X and Y

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 f_X(x)dx = \int_0^1 x^2 * \frac{2}{5}(3x+1)dx \\
&= \frac{2}{5}\left[\frac{3}{4}x^4 + \frac{x^3}{3}\right]_0^1 = \frac{2}{5}\left[\frac{3}{4}+\frac{1}{3}\right] \\
&= \frac{13}{30} \\
E(Y^2) &= \int_0^1 y^2 f_Y(y)dx = \int_0^1 y^2 * \frac{3}{5}(2y^2+1)dy \\
&= \frac{3}{5}\left[\frac{2}{5}y^5 + \frac{y^3}{3}\right]_0^1 = \frac{3}{5}\left[\frac{2}{5}+\frac{1}{3}\right] \\
&= \frac{33}{75}
\end{aligned}$$

Here $f_Y(y)$ is

$$\begin{aligned}
f_Y(y) &= \int_0^1 \frac{6}{5}(x+y^2)dx \\
&= \frac{6}{5}\left(\frac{x^2}{2} + xy^2\right)_0^1 \\
&= \frac{6}{5}\left(0.5+y^2\right) \\
&= \frac{3}{5}\left(2y^2+1\right)
\end{aligned}$$

Calculate the standard deviation of X and Y

$$\sigma_X = [E(X^2)) - (E(X))^2]^{1/2} = [(13/30) - (3/5)^2]^{1/2} = 0.271$$
$$\sigma_Y = [E(Y^2)) - (E(Y))^2]^{1/2} = [(11/25) - (3/5)^2]^{1/2} = 0.283$$

Hence the correlation coefficient,

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.01}{0.271 * 0.283} \simeq -0.131$$

Correlation is small.

Problem 3.59

a) The summation of PMF for the conditions that precipitation should be equal to 2 or more and run off more than 20 cfs gives the required probability. The values of PMF are taken from the table given in the question.
Calculate the probability that the next storm has a precipitation of 2 in. or more and runoff of more than 20 cfs.
$$P(X \geq 2,\ Y > 20) = 0.1 + 0.1 = 0.2$$

b) Given that $X = 2$, the new, reduced sample space corresponds to only the second column, where probabilities sum to $(0.15 + 0.25 + 0.10) = 0.5$(Refer to formula 3.61), not one, so all those probabiliies should now be divided by 0.5. Hence
$$\begin{aligned}
P(Y \leq 20 | X = 2) &= 0.25/0.5 + 0.1/0.5 \\
&= 0.35/0.5 \\
&= 0.7
\end{aligned}$$

c) If X and Y are statistically independent, then (say) $P(Y \geq 20)$ should be the same as $P(Y \geq 20|X = 2) = 0.7$. However,

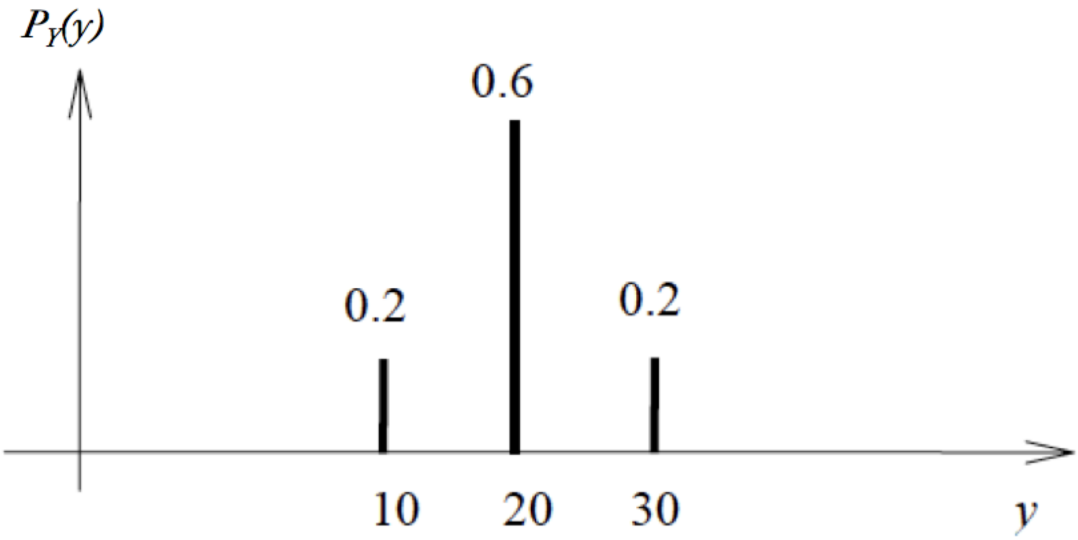
$$\begin{aligned} P(Y \geq 20) &= 0.10 + 0.25 + 0.25 + 0.0 + 0.10 + 0.10 \\ &= 0.80 \neq 0.7, \end{aligned}$$

hence X and Y are **not statistically independent.**

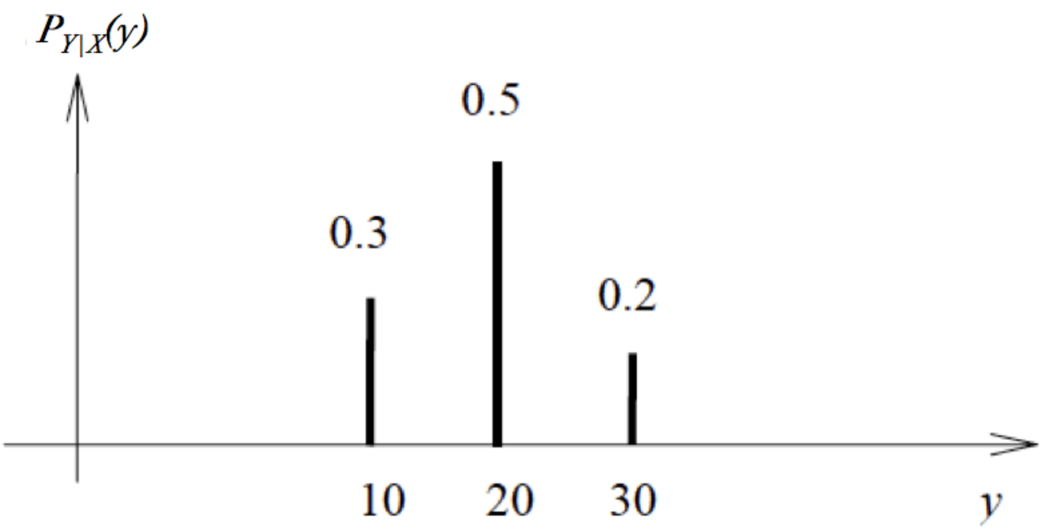
d) Summing over each row, we obtain the (unconditional) probabilities

$$P_Y(y_i) = P(Y = y_i) = \sum_{all/:x_i} p(X = x_i, Y = y_i)$$
$$P(Y = 10) = 0.20,\ P(Y = 20) = 0.60,\ P(Y = 30) = 0.20$$

Hence the margianal PMF of roundoff Y is as follows:



e) Given that X=2, we use the probabilities in the X=2 column, each multiplied by 2 so that their sum is unitiy. Hence we have $P(Y = 10|X = 2) = 0.15 * 2 = 0.30$, $P(Y = 20|X = 2) = 0.25 * 2 = 0.50$, $P(Y = 30|X = 2) = 0.10 * 2 = 0.20$, and hence the PMF plot:



f) By summing over each column, we obtain the marginal PMF of X as $P(X=1)=0.15, P(X=2)=0.5,\ P(X=3)=0.35$. With these, and results from pard d), we calculate

$$\begin{aligned}
E(X) &= P(X=1)*1+P(X=2)*2+P(X=3)*3 = 0.15*1+0.5*2+0.35*3 = 2.2 \\
Var(X) &= P(X=1)*(1-E(X))^2+P(X=2)*(2-E(X))^2+P(X=3)*(1-E(X))^2 \\
&= 0.15*(1-2.2)^2+0.5*(2-2.2)^2+0.35*(3-2.2)^2 = 0.46,\ \text{similarily} \\
E(Y) &= 0.2*10+0.6*20+0.2*30 = 20 \\
Var(Y) &= 0.2*(10-20)^2+0.6*(20-20)^2+0.2*(30-20)^2 = 40,
\end{aligned}$$

Also,

$$\begin{aligned}
E(XY) = &\sum_{all\ x,y} xyf(x,y) \\
&= 1*10*0.05+2*10*0.15+1*20*0.10+2*20*0.28 \\
&+3*20*0.25+2*30*0.10+3*30*0.10 \\
&= 45.5
\end{aligned}$$

Hence the correlation coefficient is

$$\begin{aligned}
\rho &= \frac{E(XY)-E(X)E(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\
&= \frac{45.5-2.2*20}{\sqrt{0.46*40}} \\
&\simeq 0.35
\end{aligned}$$

Problem 4.3

Let's define the nuclear, fossil, and hydroelectric power supply distributions as

$$\begin{aligned}
N &\sim N(100, 15) \\
F &\sim N(200, 40) \\
H &\sim N(400, 100)
\end{aligned}$$

and T represents the total powersupply. $T = N + F + H$

a) The total power supply for the region is the sum of the independent power supplies, and any linear combination of normal distributions is another normal distribution, so can find

$$\begin{aligned}
\mu_T &= \mu_N+\mu_F+\mu_H = 100+200+400 = 700 \\
{\sigma_T}^2 &= {\sigma_N}^2+{\sigma_F}^2+{\sigma_H}^2 = 15^2+40^2+100^2 = 11825 \\
\sigma_T &= 108.74
\end{aligned}$$

So $T \sim N(700, 108.7)$.

b) Let's define the following events:

N = normal (good) weather

E = extreme weather

S = power shortage.

We know

$$
\begin{aligned}
P(N) &= 2P(E) \text{ and } P(N) + P(E) = 1, \text{ so} \\
P(N) &= \frac{2}{3}, P(E) = \frac{1}{3} \\
P(S) &= P(S|N)P(N) + P(S|E)P(E) \\
&= P(x \leq 400)\frac{2}{3} + P(x \leq 600)\frac{1}{3} \\
&= \Phi\left(\frac{400-700}{108.74}\right)\frac{2}{3} + \Phi\left(\frac{600-700}{108.74}\right)\frac{1}{3} \\
&= (1-\Phi(2.76))\frac{2}{3} + (1-\Phi(0.92))\frac{1}{3} = (1-0.997) * \frac{2}{3} + (1-0.821) * \frac{1}{3} \\
&= 0.0615
\end{aligned}
$$

c) We apply Bayes' Theorem to find

$$
\begin{aligned}
P(N|S) &= \frac{P(S|N)P(N)}{P(S)} \\
&= \frac{(1-0.997)\frac{2}{3}}{0.0615} \\
&= 0.0314
\end{aligned}
$$

d) Define the following events:

N_u = nuclear supplies enough power

F = fossil supplies enough power

H = hydroelectric supplies enough power

$$
\begin{aligned}
P(\text{at least one fails}) &= 1 - P(\text{none fail}) \\
&= 1 - P(N_u)P(F)P(H) \\
&= 1 - P(N_u \geq 0.15 * 400)P(F \geq 0.30 * 400)P(H \geq 0.55 * 400) \\
&= 1 - \left(1-\Phi\left(\frac{60-100}{15}\right)\right)\left(1-\Phi\left(\frac{120-200}{40}\right)\right)\left(1-\Phi\left(\frac{220-400}{100}\right)\right) \\
&= 1 - \Phi(2.67)\Phi(2.00)\Phi(1.80) = 1 - 0.9962 \cdot 0.9772 \cdot 0.964 = 0.0616.
\end{aligned}
$$

Problem 4.5

a) We need to find the distribution of the settlement first, we know it is normal because it is a linear combination of normal distributions.

$$
\begin{aligned}
\mu_S &= 0.3\mu_A + 0.2\mu_B + 0.1\mu_C \\
&= 0.3 \cdot 5 + 0.2 \cdot 8 + 0.1 \cdot 7 = 3.8 \\
{\sigma_S}^2 &= 0.3^2 \cdot 1^2 + 0.2^2 \cdot 2^2 + 0.1^2 \cdot 1^2 = 0.26 \\
\sigma_S &= 0.51.
\end{aligned}
$$

So $S \sim N(3.8, 0.51)$, now we can find

$$
\begin{aligned}
P(S > 4) &= 1 - P(S \leq 4) \\
&= 1 - \Phi\left(\frac{4-3.8}{0.51}\right) = 1 - \Phi(0.3922) \quad = 1 - 0.6526 = 0.347
\end{aligned}
$$

b) If $A + B + C = 20$ then our updated equation is

$$\begin{aligned} S &= 0.3A + 0.2B + 0.1(20 - A - B) \\ &= 0.2A + 0.1B + 2 \end{aligned}$$

and we know $\rho_{AB} = 0.5$, so

$$\begin{aligned} \mu_S &= 0.2\mu_A + 0.1\mu_B + 2 \\ &= 0.2 \cdot 5 + 0.1 \cdot 8 + 2 \\ &= 3.8 \end{aligned}$$

Find the standard deviation of settlement of the building.

$$\begin{aligned} {\sigma_S}^2 &= (0.2\sigma_A)^2 + (0.1\sigma_B)^2 + 2(0.1)\rho_{AB}\sigma_A\sigma_B \\ &= 0.2^2 \cdot 1^2 + 0.1^2 \cdot 2^2 + 2 \cdot 0.1 \cdot 0.5 \cdot 1 \cdot 2 \\ &= 0.04 + 0.04 + 0.04 = 0.12 \\ \sigma_S &= 0.346 \end{aligned}$$

$$\begin{aligned} P(S > 4) &= 1 - P(S \leq 4) \\ &= 1 - \Phi\left(\frac{4 - 3.8}{0.346}\right) = 1 - \Phi(0.577) \\ &= 1 - 0.718 = 0.282. \end{aligned}$$

Problem 6.1

a) The sample mean and sample variance are given by

$$\begin{aligned} \overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ &= \frac{1}{9}(82 + 75 + 95 + 90 + 88 + 92 + 78 + 85 + 80) \\ &= 85 \end{aligned}$$

Calculate sample variance (s^2) of individual pile capacity.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Here, sample value is x_i and mean sample is $\bar{x}$.
Substitute 9 for n and 85 for $\bar{x}$

$$\begin{aligned} s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \\ &= \frac{1}{9-1}((82-85)^2 + (75-85)^2 + (95-85)^2 + (90-85)^2 + (88-85)^2 \\ &+ (92-85)^2 + (78-85)^2 + (85-85)^2 + (80-85)^2) \\ &= \frac{1}{9-1}(3^2 + 10^2 + 10^2 + 5^2 + 3^2 + 7^2 + 7^2 + 0^2 + 5^2) \\ &= \frac{366}{8} = 45.75 \\ s &= 6.764 \end{aligned}$$

b) Calculate the value of test statistics (t).
Consider that null hypthesis of mean pile capacity is 80 tons

$$H_0 : \mu = 80 \text{ tons}$$
$$H_A : \mu < 80 \text{ tons}$$

We estimate the test statistic to be

$$t = \frac{\overline{x} - 80}{s/\sqrt{n}} = \frac{85 - 80}{6.76/3} = 2.217$$

Calculate the degree of freedom(f)

$$f = 9 - 1 = 8$$

From the critical values of $t-$distribution at confidence level table, select the critical value of t based on degrees of freedom of 8 at 5% significance level as -1.8595. Therefore, the test statistic value is $2.217 > -1.8595$ is outside of rejection region. Hence, null hypothesis and pile capacity remain acceptable.

c) Calculate the probability parameter $\left(k_{\alpha/2}\right)$
Consider the standard deviation of the population is $\sigma = s_0$

$$k_{\alpha/2} = \Phi^{-1}(\alpha/2)$$

Here, parameter $1 - \alpha$=0.98

$$\begin{aligned} k_{(1-0.98)/2} &= \Phi^{-1}\left((1 - 0.98)/2\right)\right) \\ k_{0.01} &= \Phi^{-1}(0.01) = -\Phi^{-1}(0.99) = -k_{0.99} \\ k_{0.01} &= -2.33 \end{aligned}$$

The 98% confidence interval is given by

$$\begin{aligned} < \mu_x >_{0.98} &= \left(\overline{x} + k_{0.01} \cdot \frac{\sigma_x}{\sqrt{n}}, \overline{x} - k_{0.01} \cdot \frac{\sigma_x}{\sqrt{n}}\right) \\ &= \left(85 - 2.33 \cdot \frac{6.764}{\sqrt{9}}, \overline{x} + 2.33 \cdot \frac{6.764}{\sqrt{9}}\right) \\ &= (79.74, 90.25) \end{aligned}$$

d) Calculate confidence interval when the population variance is unknown.
$t_{\alpha/2,n-1} = t(\alpha/2, n - 1)$
Substitute (1-0.98) for α, and 9 for n

$$\begin{aligned} t_{(1-0.98)/2,9-1} &= t((1 - 0.98)/2, 9 - 1) \\ t_{0.01,8} &= t(0.01, 8) = -t(0.99, 8) \end{aligned}$$

From the critical values of $t-$distribution table, select critical value of t based on degrees of freedom of 9 and parameter of 0.99 as 2.8965.
$t_{0.01,8} = -2.8965$, $t_{0.99,8} = 2.8965$ using $d.o.f = 8$ and $p = 0.990$ from the appendix.
Then,

$$\begin{aligned} < \mu_x >_{0.98} &= \left(\overline{x} + t_{0.01,8} \cdot \frac{\sigma_x}{\sqrt{n}}, \overline{x} - t_{0.01,8} \cdot \frac{\sigma_x}{\sqrt{n}}\right) \\ &= \left(85 - 2.8965 \cdot \frac{6.764}{\sqrt{9}}, \overline{x} + 2.8965 \cdot \frac{6.764}{\sqrt{9}}\right) \\ &= (78.4, 91.5) \end{aligned}$$

Problem 6.5

a) Let x be the concrete strength

$$\bar{x} = \frac{\sum_{i=1}^{5} x_i}{5} = \frac{18360}{5} = 3672, \quad s_x = \sqrt{\frac{\sum_{i=1}^{5}(x_i - \bar{x})^2}{5-1}} = \sqrt{\frac{589778}{4}} = 384$$

Calculate the statistic distribution parameter ($t_{\alpha/2,n-1}$)

$$\begin{aligned} t_{(1-0.90)/2,5-1} &= t((1-0.90)/2, 5-1) \\ t_{0.05,4} &= t(0.05,4) = -t(0.95,4) \\ &= -2.1318 \end{aligned}$$

Calculate the 90% two sided confidence interval ($(\mu_x)_{0.90}$) of the mean concrete strength.

$$<\mu_x>_{0.90}= \left(\bar{x} + t_{0.05,4}\frac{s_x}{\sqrt{n}}, \bar{x} - t_{0.05,4}\frac{s_x}{\sqrt{n}}\right)$$

Here, sample mean is $\bar{x}$

$$\begin{aligned} <\mu_x>_{0.90} &= \left(3672 - 2.1318\frac{384}{\sqrt{5}}, 3672 + 2.1318\frac{384}{\sqrt{5}}\right) \\ &= (3305, 4038) \end{aligned}$$

b) Express the half width of confidence interval of the mean concrete strength.
Consider that mean sample concrete strength of confidence interval is ± 300 psi.

$$\begin{aligned} t_{1-\alpha/2,n-1}\left(\frac{s_x}{\sqrt{n}}\right) &= 300 \text{ psi} \\ t_{0.95,4}\left(\frac{384}{\sqrt{5}}\right) &= 300 \\ &= \frac{300}{171.73} \\ &= 1.747 \end{aligned}$$

From the critical values of $t-$distribution table, select critical value of t based on degree of freedom of 4 and parameter of 0.900 as 1.5332 and degree of 4 and parameter of 0.950 as 2.1318.

$$\begin{aligned} t_{0.9,4} &= 1.533, \\ t_{0.95,4} &= 2.132, \end{aligned}$$

and $t_{1-\alpha/2,4}$ increases as $\alpha/2$ gets smaller in general.

Hence we may use linear interpolation to get an approximate answer: over such a small "t" range (from t=1.533 to t=2.132), we treat "y"(i.e. $\alpha/2$) as a linear function of t which decreases linearly, with slope

$$\begin{aligned} 0.1 &= m_0 \cdot 1.533 + b_0 \\ 0.005 &= m_0 \cdot 2.132 + b_0 \\ \rightarrow m_0 &= \frac{0.1 - 0.05}{1.533 - 2.132} = -0.083472454 \simeq -0.0835 \\ b_0 &= 0.1 + 0.0835 \cdot 1.533 = 0.228 \end{aligned}$$

Express the confidence level (α) associate with specific interval based measurement.

$$\frac{\alpha}{2} = m_0 t + b_0$$

For $t = 1.74$, we obtain $\alpha = 0.16542$
Hence the confidence level is $1 - \alpha = 1 - 0.06542 \simeq 83.6\%$

Problem 6.6

a) Calculate the two sided 99% confidence interval $< \mu_x >_{0.99}$ for the mean weight of trailer trucks. Note that, $\alpha = 1 - 0.99 = 0.01$. Then,

$$\begin{aligned} k_{\alpha/2} &= \Phi^{-1}(\alpha/2) \\ k_{0.005} &= \Phi^{-1}(0.005) \\ &= -\Phi^{-1}(0.995) \\ &= 2.58 = k_{1-\frac{\alpha}{2}} = k_{0.995} \end{aligned}$$

$$< \mu_x >_{0.99} = \left(\bar{x} + k_{0.005}\frac{\sigma}{\sqrt{n}}, \bar{x} - k_{0.005}\frac{\sigma}{\sqrt{n}}\right) = \left(\bar{x} - k_{0.995}\frac{\sigma}{\sqrt{n}}, \bar{x} + k_{0.995}\frac{\sigma}{\sqrt{n}}\right)$$

Replacing $n = 30, \bar{x} = 12.5\, tons, \sigma = 3\, tons.$

$$\begin{aligned} < \mu_x >_{0.99} &= \left(12.5 - 2.58 * \frac{3}{\sqrt{30}}, 12.5 + 2.58 * \frac{3}{\sqrt{30}}\right) \\ &= (12.5 - 2.826, 12.5 + 2.826) \\ &= (9.674, 15.326)\, tons \end{aligned}$$

b) Obtain the number of additional trucks needed so that the true mean weight of trucks is within $\pm 1\, ton$ with 99% confidence.

$$\begin{aligned} k_{0.995}\left(\frac{\sigma}{\sqrt{n}}\right) &= 1\, ton \\ 2.58 * \frac{3}{\sqrt{n}} &= 1 \\ n = (2.58 * 3)^2 &= 59.9 \\ &\simeq 60 \end{aligned}$$

Therefore, (60-30) or 30 additional observations of truck weights would be required.

Problem 6.8

a) Calculate the sample mean of daily dissolved oxygen concentration(DO).
Consider that null hypthesis of mean EPA is 2.0 mg/l

$$\begin{aligned} H_0 &: \mu = 2.0\,\text{mg/l} \\ H_A &: \mu < 2.0\,\text{mg/l} \end{aligned}$$

$$\begin{aligned} \bar{x} &= \frac{\sum_{i=1}^{n} x_i}{n} \\ &= \frac{1.8 + 2 + 2.1 + 1.7 + 1.2 + 2.3 + 2.5 + 2.9 + 1.9 + 2.2}{n} \\ &= \frac{20.6}{n} \\ &= \frac{20.6}{10} \\ &= 2.06\, mg/l \end{aligned}$$

Calculate the sample variance s_x^2 of oxygen concentration.

$$s_x^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} = \frac{1.940}{9} = 0.215$$
$$s_x = 0.464\, mg/l$$

Calculate the value of the test statistics (t).

$$\begin{aligned} t &= \frac{\bar{x} - \mu}{s_x/\sqrt{n}} \\ &= \frac{2.06 - 2.00}{0.464/\sqrt{10}} \\ &= 0.409 \end{aligned}$$

Here, concentration of environmental protection agency is μ.
Calculate the degrees of freedom (f).

$$f = n - 1 = 10 - 1 = 9$$

From the critical values of $t-$distribution at confidence level table, select the critical value of t based on degrees of freedom of 9 at 5% significance level as 1.833. Therefore, the test statistic value is 0.409¡1.833 is outside of rejection region. Hence, stream quality satisfies the EPA standard at significance level of 5%.

b) Estimate the values of parameter μ and σ of the normal distribution $N(\mu, \sigma)$.
We need to determine the confidence interval. In other words, through the method of mements one can just take $\bar{x}$ and s as the estimates of the mean and teh standard deviation. And here, let's assume it as 95% or 99% which are two of a particular choice of estimator (just two of many).
Calculate the statistic distribution parameter, $t_{1-\alpha,n-1}$
From Table A.3, with $f = 9$ d.o.f., we obtain the critical values as followings:
95% confidence interval $t_{0.025,9} = -2.2622 = -t_{0.975,9}$
99% confidence interval $t_{0.005,9} = -3.2498 = -t_{0.995,9}$
Then,

$$< \mu_X >_{0.95} = \left(2.06 - 2.2622\frac{0.464}{\sqrt{10}}; 2.06 + 2.2622\frac{0.464}{\sqrt{10}}\right) = (1.728; 2.392)\text{mg}$$
$$< \mu_X >_{0.99} = \left(2.06 - 3.2498\frac{0.464}{\sqrt{10}}; 2.06 + 3.2498\frac{0.464}{\sqrt{10}}\right) = (1.583; 2.537)\text{mg}$$

Calculate the value of distribution parameter.

$$\begin{aligned} (\sigma_X)_{0.95} &= (\sigma_X)_{0.99} \\ &= \sqrt{s_x^2} \\ &= \sqrt{0.464^2} \\ &= 0.464\ \text{mg/l} \end{aligned}$$

c)

$$\begin{aligned} < \mu_x >_{0.95} &= \left(\bar{x} + t_{0.025,9}\frac{S_x}{\sqrt{n}}, \bar{x} - t_{0.025,9}\frac{S_x}{\sqrt{n}}\right) \\ &= \left(2.06 - (2.2622)\frac{0.464}{\sqrt{10}}, 2.06 + (2.2622)\frac{0.464}{\sqrt{10}}\right) \\ &= (1.728, 2.392)\ mg/l \end{aligned}$$