MAE108 S2014 - Homework 5 Solutions

Problem 3.15

We are given $\nu = 1/5000$ failures/hr

a) We let $\nu * (t = 2500) = 1/2$. Then we say

$$P(X_t \geq 1) = 1 - P(X_t = 0) = 1 - \frac{(\nu t)^0}{0!} e^{-\nu t} = 1 - e^{-1/2} = 0.3935$$

b) The number of airplanes with mechanical failures Y with $p = 1 - e^{-1/2}$ is given

$$\begin{aligned} P(Y_{10} \leq 2) &= P(Y_{10} = 0) + P(Y_{10} = 1) + P(Y_{10} = 2) \\ &= \binom{10}{0} p^0 (1-p)^{(10-0)} + \binom{10}{1} p^1 (1-p)^{(10-1)} + \binom{10}{2} p^2 (1-p)^{(10-2)} \\ &= e^{-5} + 10(1 - e^{-1/2}) e^{9/2} + 45(1 - e^{-1/2})^2 e^{-4} = 0.1781 \end{aligned}$$

c) We want

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{(\nu t')^0}{0!} e^{-\nu t'} = 1 - e^{-\nu t'} \leq 0.05 \rightarrow t' \leq -ln(19/20)/\nu = 256.5$$

There should be an inspection every 256.6 hours or less.

Problem 3.17

a) Let N be the number of poor air quality periods during the next 4.5 months; N follows a Poisson process with mean value, $\lambda = \nu$t= (1/month)(4.5 months) =4.5, hence

$$\begin{aligned} P(N \leq 2) &= P(N = 0) + P(N = 1) + P(N = 2) \\ &= \frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} = e^{-4.5}(1 + 4.5 + 4.5^2) \\ &= 0.174 \end{aligned}$$

b) Since only 0.1 of poor quality periods have hazadous levels, the "hazadous" periods (H) must occur at a mean rate of 1 per month*0.1=0.1 per month, hence, over 3 months, H has the mean

$$\lambda_H = (0.1) * 3 = 0.3$$

$$\rightarrow P(\text{ever hazardous}) = 1 - P(H = 0) = 1 - \frac{(0.3)^0}{0!} e^{-0.3} = 0.259$$

Problem 3.19

a) Mean rate of accident: $\nu = 1/3$ accident per year, $\lambda_5 = \nu * t = 1/3 * 5$

$$P(N = 0 \text{ in 5 years}) = \frac{\lambda_5^0}{0!} e^{-\lambda_5} = e^{-5/3} = e^{-1.667} = 0.1889$$

b) Let N the number of fatal accidents over 3 years
Mean rate of fatal accident: $\nu_D = 1/3 * 0.05 = 1/60 = 0.01667$ per year, $\lambda_{D3} = \nu_D * t = 1/60 * 3 = 0.05$

$$\begin{aligned} P(N \geq 1) &= 1 - P(N = 0) \\ &= 1 - \frac{\lambda_{D3}^0}{0!} e^{-\lambda_{D3}} \\ &= 1 - e^{-0.05} \qquad = 0.0488 \end{aligned}$$

Problem 3.21

We are given $\nu = 3$ accidents/year= 3/12 accidents/month

a) Let X be the number of accidents
in two months $\lambda_2 = \nu * 2$ months

$$\lambda_2 = \frac{3}{12\text{months}} * 2\text{months} = 0.5$$

in four months $\lambda_4 = \nu * 4$ months

$$\lambda_4 = \frac{3}{12\text{months}} * 4\text{months} = 1$$

Calculate the probability of one accident in two months

$$P(X = 1) = \frac{(\lambda_2)^1}{1!} e^{-\lambda_2} = \frac{0.5}{1} e^{-0.5} = 0.303$$

and the probability of two accidents in four months

$$P(X = 2) = \frac{(\lambda_4)^2}{2!} e^{-\lambda_4} = \frac{1}{2!} e^{-1} = 0.184$$

So to probability of one accident in two months and probability of exactly two accidents in four months is not the same. In other words, they are not a uniform distibuted.

b) Let F be the number of fatal accidents Calculate the mean rate of fatal accidents

$$\begin{aligned} \nu_F &= \text{accident per month * percentage of fatal accidents} \\ &= \frac{3}{12} * \frac{20}{100} \\ &= 0.05 \text{ per month} \end{aligned}$$

Calculate the average number of fatal accidents in two months.

$$\lambda_F = \nu_F * 2 \text{ months} = 0.05 * 2 = 0.1$$

Calculate the probability of the fatal accident in two months.

$$\begin{aligned} P(F) &= 1 - P(F = 0) \\ &= 1 - \frac{\lambda_F^0}{0!} e^{-\lambda_F} \\ &= 1 - \frac{0.1^0}{1} e^{-0.1} \\ &= 1 - 0.904 \\ &= 0.095 \end{aligned}$$

Problem 3.23

Let the area of the panel t, $t = l * w$, where t, l and w are area, length and width.
$t = 3m * 5m = 15m^2$
Calculate the average number of flaws in t when the mean flaw rate(ν) is $1/50m^2$

$$\lambda_t = \nu * t = 1/50m^2 * 15m^2 = 0.3$$

a) The panel is replaced when it contains two or more flaws. And let X be the number of flaws in a panel. $P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}$ because it follows Poisson process

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\lambda_t^0}{0!}e^{-0.3} - \frac{\lambda_t^1}{1!}e^{-0.3} = 1 - 0.741 - 0.222 = 0.037$$

b) Calculate the average number of replacements in 100 panels. Let N be the average number of panels to be replaced. Let C_R be the replacement cost.

$$\begin{aligned} N &= P(X \geq 2) * 100 \text{ panels} \\ &= 0.037 * 100 = 3.7 \text{ panels} \\ C_R &= N * \text{cost per replacement} \\ &= 3.7\text{panels} * \$5000 \\ &= \$18,500 \end{aligned}$$

c) The mean flaw rate for higher grade panels, ν is $1/80m^2$ and same process as a) and b)
$t = 3m * 5m = 15m^2$
Calculate the average number of flaws in t when the mean flaw rate(ν) is $1/80m^2$

$$\lambda_t = \nu * t = 1/80m^2 * 15m^2 = 0.1875$$

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{\lambda_t^0}{0!}e^{-0.3} - \frac{\lambda_t^1}{1!}e^{-0.3} = 1 - 0.829 - 0.155 = 0.0155$$

$$\begin{aligned} N &= P(X \geq 2) * 100 \text{ panels} \\ &= 0.0155 * 100 = 1.55 \text{ panels} \\ C_R &= N * \text{cost per replacement} \\ &= 1.55 \text{ panels} * \$5,100 \\ &= \$7,920 \end{aligned}$$

The initial cost of the old panel type is taken as C. The initial cost per higher grade panel is 100 more than the old panel type. So the total initial cost of the higher grade panel is obtained as $C + \$100$ per panel $*100$ panels

(a) total expeted cost of old type panel, $EC1$

$$EC1 = \text{initial cost} + \text{replacement cost} = C + \$18,500$$

(b) total expected cost of higher grade panel, $EC2$

$$\begin{aligned} EC2 &= \text{initial cost} + \text{replacement cost} \\ &= (C + \$100 * 100) + \$7,290 = C + \$17,290 \end{aligned}$$

Comparing $EC1$ and $EC2$, the total expected cost of higher grade panels is lower. Therefore, the higher grade panels are recommended.

Problem 3.27

a) Let X be the total number of excavations along the pipeline over the next year; X has a Poisson distribution with a mean occurence rate $= \nu = 1/50$ exacavations per mile. Then, $\lambda_{100} = \nu * t = 1/50$ excavations/miles $*100$ miles=2 (excavations)

$$\begin{aligned} P(\text{at least two excavations}) = P(X \geq 2) &= 1 - P(X=0) - P(X=1) \\ &= 1 - \frac{\lambda_{100}^0}{0!} e^{-\lambda_{100}} - \frac{\lambda_{100}^1}{1!} e^{-\lambda_{100}} \\ &= 1 - 0.1353 - 0.2706 \\ &= 0.594 \end{aligned}$$

b) For each excavation that takes place, the pipeline has 0.4 probability of getting damaged and hence the probability of having no damage space(p')=(1-0.4)=0.6

$$P(\text{any damage to pipeline}|X=2) = 1 - P(\text{no damge}|X=2) = 1 - 0.6^2 = 0.64$$

Alternative method

Let D_i denote the event "damage to pipeline in i-th excavation"; the desired probability is

$$\begin{aligned} P(D_1 \cup D_2) &= P(D_1) + P(D_2) - P(D_1 D_2) \\ &= P(D_1) + P(D_2) - P(D_1|D_2)P(D_2) \\ &= P(D_1) + P(D_2) - P(D_1)P(D_2) \\ &= 0.4 + 0.4 - 0.4^2 = 0.64 \end{aligned}$$

c) Any number x of excavations could take place, but there must be no damage no matter what the x value is. Denoting $\lambda = \nu * 100 = 2$, we have the total probability

$$\begin{aligned} P(\text{no damage}) &= \sum_{x=0}^{\infty} P(\text{no damage}|x \text{ excavation}) * P(x \text{ excavations}) \\ &= \sum_{x=0}^{\infty} (p')^x e^{-\lambda} \frac{\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} (0.6)^x e^{-2} \frac{2^x}{x!} \\ &= e^{-2} \sum_{x=0}^{\infty} \frac{(0.6*2)^x}{x!} ...(*) \\ &= e^{-2} * e^{1.2} \\ &= e^{-0.8} \\ &= 0.449 \end{aligned}$$

$$(*)... \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

Problem 3.32

a) The mean rate of defects, ν_D is 1/200m. Let D denote defects and the defects that remain after inspection is denoted by R.
Calculate mean rate of defects that remain in the system after inspection $\nu_R = \nu_D * P(D)$

where ν_R: the mean rate of defects that remain after inspection, $P(D)$: the probability that the defect remains after inspection.

$$\begin{aligned}\nu_R &= 1/200m * 0.2\\ &= 0.001 \text{ per meter}\end{aligned}$$

b) Calculate the average number of occurences of defects. $\lambda_{3000} = \nu_R * t = 0.001 * 3000 = 3$
Calculate the probability that there are the more than two defects.

$$\begin{aligned}P(D > 2) &= 1 - P(D = 0) - P(D = 1) - P(D = 2)\\ &= 1 - \frac{\lambda_{3000}^0}{0!}e^{-\lambda_{3000}} - \frac{\lambda_{3000}^1}{1!}e^{-\lambda_{3000}} - \frac{\lambda_{3000}^2}{2!}e^{-\lambda_{3000}}\\ &= 1 - \frac{3^0}{0!}e^{-3} - \frac{3^1}{1!}e^{-3} - \frac{3^2}{2!}e^{-3}\\ &= 1 - 0.0497 - 0.149 - 0.224\\ &= 0.577\end{aligned}$$

c) Let $p = P(D)$.
$\nu_R = \nu_D * p = 1/200\text{m} * p = 0.005p$ per meter
Calculate the average number of occurrence of defects for 1,000m.
$\lambda = \nu_R * t = 0.05p$ per meter $* 1000$ meters $= 5p$
The system is required to achieve 95% probability free of deects.

$$\begin{aligned}P(D = 0) &= 95\%\\ \frac{\lambda^0}{0!}e^{-\lambda} &= 0.95\\ \frac{(5p)^0}{1}e^{-5p} &= 0.95\\ -5p &= \ln 0.95\\ p &= \frac{-0.051}{-5}\\ &= 0.0103\\ &= 1\%\end{aligned}$$

Problem 3.34

a) The number of defects should be zero for all 20 bars to pass the test.
And let $N = 1000$: total size of the rebars, $m = 20$: the number of defective rebars in N, n :the number of sample rebars taken from the total rebar(N), and x: the number of defective rebars in our subsample of 20 elements(m)

all 20 rebars will pass test means no defective rebars in the sample(x=0).

$$
\begin{aligned}
P(X=0) &= \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}\Big|_{x=0} \\
&= \frac{\binom{20}{0}\binom{1000-20}{20-0}}{\binom{1000}{20}} \\
&= \frac{\frac{20!}{0!(20-0)!}\frac{980!}{20!(980-20)!}}{\frac{1000!}{20!(1000-20)!}} \\
&= \frac{1 * 2.2575 * 10^41}{3.3947 * 10^41} \\
&= 0.665
\end{aligned}
$$

b) Calculate the probability that at least two bars will fail

$$
\begin{aligned}
P(X \geq 2) &= 1 - P(x=0) - P(x=1) \\
&= 1 - \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}\Big|_{X=0} - \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}\Big|_{X=1} \\
&= 1 - \frac{\binom{20}{0}\binom{1000-20}{20-0}}{\binom{1000}{20}} - \frac{\binom{20}{1}\binom{1000-20}{20-1}}{\binom{1000}{20}} \\
&= 1 - \frac{\frac{20!}{0!(20-0)!}\frac{980!}{20!(980-20)!}}{\frac{1000!}{20!(1000-20)!}} - \frac{\frac{20!}{1!(20-1)!}\frac{980!}{19!(980-19)!}}{\frac{1000!}{20!(1000-20)!}} \\
&= 1 - \frac{1 * 2.2575 * 10^{41}}{3.3947 * 10^{41}} - \frac{20 * 4.6982 * 10^{39}}{3.3947 * 10^{41}} \\
&= 1 - 0.665 - 0.277 \\
&= 0.058
\end{aligned}
$$

c) The company assures 90% of quality which could contain 2% defective rebars. So there will be samples containing 2% defectives with 90% certainty.

$$
\begin{aligned}
1 - P(X \geq 1) &= 1 - 0.9 \\
P(X=0) &= \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \\
&= \frac{\binom{20}{0}\binom{1000-20}{n-0}}{\binom{1000}{n}} \\
0.10 &= \frac{\frac{20!}{0!(20-0)!}\frac{980!}{n!(980-n)!}}{\frac{1000!}{n!(1000-n)!}} \\
&= 1 * \frac{980!}{n!(980-n)!} * \frac{n!(1000-n)!}{1000!} ...(**) \\
&= \frac{1}{(980-n)!} * \frac{(1000-n)!}{8.258 * 10^{59}}
\end{aligned}
$$

It's not easy to calculate this value but could get the value using online website.(www.wolframalpha.com). Once you type the equation which is at (**) into the box, you can get the value from the plot. Put cursor on the red point and see the value(107.712). So 108 rebar is required.