MAE108 S2014 - Homework 4 Solutions

Problem 2.56

Let's define the following events:
A = a site contains an anomaly.
T_1 = the first geophysical technique detects an anomaly.
We know $P(A) = 0.3, P(T_1|A) = 0.5$, and $P(T_1|\overline{A}) = 0$.

a) The probability of an anomaly given failure of the first geophysical technique is

$$P(A|\overline{T}_1) = \frac{P(A)}{P(\overline{T}_1)} P(\overline{T}_1|A) = \frac{P(A)}{1 - P(T_1)}(1 - P(T_1|A))$$
$$= \frac{0.3}{1 - 0.15}(1 - 0.5) = 0.176$$

where

$$P(T_1) = P(T_1|A)P(A) + P(T|\overline{A})P(\overline{A}) = 0.5 * 0.3 + 0 * 0.7 = 0.15.$$

b) We have a new event T_2 = the second geophysical technique detects an anomaly. For the rest of this question we will assume that T_1 occured, and we will omit it from our conditional statements. So our updated probabilities and new information gives us $P(A) = 0.176, P(T_2|A) = 0.8$, and $P(T_2|\overline{A}) = 0$.

i) The engineer's new confidence that the site is free of any abnormality is given by

$$P(\overline{A}|\overline{T}_2) = \frac{P(\overline{A})}{P(\overline{T}_2)} P(\overline{T}_2|\overline{A})$$
$$= \frac{(1 - P(A))(1 - P(T_2|\overline{A}))}{1 - [P(T_2|A)P(A) + P(T_2|\overline{A})P(\overline{A}))]} = \frac{(1 - 0.176)(1 - 0)}{1 - [0.8 * 0.176]} = 0.959.$$

ii) We define S = the event that the foundation is safe. The following quantities are given: $P(S|\overline{A}) = 0.9999, P(S|A) = 0.80$. We now assume that both T_1 and T_2 occurred and exclude both from conditionals. We are looking for

$$P(\overline{S}) = 1 - (P(S|A)P(A) + P(S|\overline{A})P(\overline{A}))$$
$$= 1 - (0.80 * (1 - 0.959) + 0.9999 * 0.959) = 0.008.$$

iii) The expected loss due to foundation failure is given by

$$\text{Expected loss} = \text{(probability of failure)} * \text{(failure loss)}$$
$$= (1 - P(S)) * \text{(failure loss)} = (1 - 0.992)(1000000) = \$8300.$$

If the site is verified as anomaly free, then the amount saved in expected loss is

$$\text{Amount saved} = 8300 - P(\overline{S}|\overline{A}) * \text{(failure loss)}$$
$$= 8300 - (1 - P(S|\overline{A})) * \text{(failure loss)}$$
$$= 8300 - (1 - 0.9999) * 1000000 = \$8200.$$

Problem 2.58

We will define the following events:

L = an earthquake has low intensity
M = an earthquake has medium intensity
H = an earthquake has high intensity
P = a building is poorly constructed
W = a building is well constructed.
D = a building is damaged from an earthquake.

We know $P(P) = 0.2, P(W) = 0.8, P(D|LP) = 0.10, P(D|MP) = 0.50, P(D|HP) = 0.90, P(D|LW) = 0, P(D|MW) = 0.05$, and $P(D|HW) = 0.20$.

From the relative likelihoods of 15:4:1 we know

$$P(L) + P(M) + P(H) = 1,\ P(L) = 15 * P(H),\ P(M) = 4 * P(H)$$
$$(1 + 4 + 15)P(H) = 1 \implies P(H) = 0.05.$$

So we also know $P(H) = 0.05, P(M) = 0.20$, and $P(L) = 0.75$.

a) The probability that a well-constructed building will be damaged during an earthquake is

$$P(D|W) = P(D|LW)P(L|W) + P(D|MW)P(M|W) + P(D|HW)P(H|W)$$

and because the quality of the building is independent of the earthquake's intensity,

$$= P(D|LW)P(L) + P(D|MW)P(M) + P(D|HW)P(H)$$
$$= 0 * 0.75 + 0.05 * 0.20 + 0.20 * 0.05 = 0.02.$$

b) The proportion of buildings damaged by the earthquake is

$$P(D) = P(D|W)P(W) + P(D|P)P(P)$$
$$= 0.02 * 0.8 + 0.22 * 0.2 = 0.06$$

where

$$P(D|P) = P(D|LP)P(L) + P(D|MP)P(M) + P(D|HP)P(H)$$
$$= 0.10 * 0.75 + 0.50 * 0.20 + 0.90 * 0.05 = 0.22.$$

c) The probability that a building is poorly constructed given that it is damaged after an earthquake is

$$P(P|D) = \frac{P(P)}{P(D)} P(D|P) = \frac{0.2}{0.06} * 0.22 = 0.73.$$

Problem 2.62

We will define the following events:

A = concrete has poor aggregates.
W = concrete has poor workmanship.

We know $P(A) = 0.2, P(W|A) = 0.3$, and $P(A|W) = 0.15$.

a) The probability of poor workmanship is

$$\begin{aligned} P(W) &= \frac{P(W|A)}{P(A|W)} P(A) \\ &= \frac{0.3}{0.15} * 0.2 = 0.4. \end{aligned}$$

b) The probability of at least one of the causes of defect is

$$\begin{aligned} P(A \cup W) &= P(A) + P(W) - P(AW) \\ &= P(A) + P(W) - P(W|A)P(A) \\ &= 0.2 + 0.4 - 0.3 * 0.2 = 0.54. \end{aligned}$$

c) The probability that only one of the two causes of defect is

$$\begin{aligned} P(A \cup W) - P(AW) &= P(A \cup W) - P(W|A)P(A) \\ &= 0.54 * 0.3 - 0.2 = 0.48. \end{aligned}$$

d) We define a new event D = the concrete is defective. We have new information that $P(D|A\overline{W}) = 0.15, P(D|\overline{A}W) = 0.20, P(D|AW) = 0.80$, and $P(D|\overline{A}\,\overline{W}) = 0.05$.

The probability of defective concrete is

$$P(D) = P(D|AW)P(AW) + P(D|A\overline{W})P(A\overline{W}) + P(D|\overline{A}W)P(\overline{A}W) + P(D|\overline{A}\,\overline{W})P(\overline{A}\,\overline{W})$$

where

$$\begin{aligned} P(AW) &= P(W|A)P(A) = 0.3 * 0.2 = 0.06 \\ P(A\overline{W}) &= (1 - P(W|A))P(A) = (1 - 0.3) * 0.2 = 0.14 \\ P(\overline{A}W) &= (1 - P(A|W))P(W) = (1 - 0.15) * 0.4 = 0.34 \\ P(D|\overline{A}\,\overline{W}) &= 1 - P(A \cup W) = 1 - 0.54 = 0.46. \end{aligned}$$

So

$$P(D) = 0.80 * 0.06 + 0.15 * 0.14 + 0.20 * 0.34 + 0.05 * 0.46 = 0.16.$$

e) The probability that defective concrete is caused by both poor aggregates and poor workmanship is

$$P(AW|D) = \frac{P(AW)}{P(D)} P(D|AW) \quad = \frac{0.06}{0.16} * 0.8 = 0.3.$$

Problem 2.65

Let's define the following events:

A = company A discovers oil.

B = company B discovers oil.

C = company C discovers oil.

We know $P(A) = 0.4, P(B) = 0.6, P(C) = 0.2$, and $P(A|B) = 1.2 * 0.48$. C is statistically independent of B or A.

a) The probability of that oil will be discovered in the area by one or more of the three companies is

$$P(A \cup B \cup C) = 1 - P(\bar{A}\bar{B}\bar{C}) = 1 - P(\bar{A}\bar{B})P(\overline{C})$$

where

$$\begin{aligned} P(\bar{A}\bar{B}) &= 1 - P(A \cup B) = 1 - P(A) - P(B) + PA|B)P(B) \\ &= 1 - 0.4 - 0.6 + 0.48 * 0.6 = 0.288 \end{aligned}$$

Hence

$$P(A \cup B \cup C) = 1 - 0.288 * 0.8 = 0.77.$$

b) The probability of that oil will be discovered by Company C given that oil is discovered in the area is

$$P(C|A \cup B \cup C) = \frac{P(C(A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(C)}{P(A \cup B \cup C)} = \frac{0.2}{0.77} = 0.26.$$

c) The probability that only one of the threes companies will discover oil in the area is

$$P(A\bar{B}\bar{C} \cup \bar{A}B\bar{C} \cup \bar{A}\bar{B}C) = P(A\bar{B}\bar{C}) + P(\bar{A}B\bar{C}) + P(\bar{A}\bar{B}C)$$

where

$$\begin{aligned} P(\bar{C}\bar{B}A) &= P(\bar{C}|\bar{B}A)P(\bar{B}|A)P(A) = P(\bar{C}) * [1 - P(B|A)] * P(A) \\ &= 0.8 * 0.28 * 0.40 = 0.0896 \\ P(\bar{C}B\bar{A}) &= P(\bar{C}|B\bar{A})P(B|\bar{A})P(\bar{A}) = P(\bar{C})\frac{P(\bar{A}|B)P(B)}{P(\bar{A})}P(\bar{A}) \\ &= [1 - P(C)] * [1 - P(A|B)] * P(B) = 0.8 * 0.52 * 0.6 = 0.2496 \\ P(C\bar{B}\bar{A}) &= P(C|\bar{B}\bar{A})P(\bar{B}|\bar{A})P(\bar{A}) = P(C)P(\bar{B}|\bar{A})P(\bar{A}) = P(C)P(\bar{A}\bar{B}) \\ &= 0.2 * 0.288 = 0.0576 \end{aligned}$$

hence,

$$P(A\bar{B}\bar{C} \cup \bar{A}B\bar{C} \cup \bar{A}\bar{B}C) = 0.0896 + 0.2496 + 0.0576 = 0.397.$$

Problem 3.9

Let's define the event

$S =$ the maximum load on a structure (in tons).

a) The PDF is necessary to produce the mode and mean value of S. Differentiating the CDF gives the PDF,

$$f_S(s) = \begin{cases} 0 & \text{for } s < 0 \\ -\frac{s^2}{288} + \frac{s}{24} & \text{for } 0 < s \le 12 \\ 0 & \text{for } s > 12 \end{cases}$$

The mode $\tilde{s}$ is where f_S has a maximum, hence setting its derivative to

$$\begin{aligned} f'_S(\tilde{s}) &= 0 \\ -\frac{\tilde{s}}{144} + \frac{1}{24} &= 0 \\ \tilde{s} &= 6 \end{aligned}$$

the mode, $\tilde{s}$, is equal to **6**.
The mean value is

$$\mu_S = \int_{-\infty}^{\infty} s f_S(s)\, ds = \int_0^{12} \frac{s^2}{24} - \frac{s^3}{288}\, ds = \left[\frac{s^3}{3*24} - \frac{s^4}{4*288}\right]_0^{12} = 6$$

b) We have a new event lets define it $R =$ the strength of the structure.
From the PMF it is clear that $P(R = 10) = 0.7$, and $P(R = 13) = 0.3$. Dividing the sample space into two regions $R = 10$ and $R = 13$, the total probability of failure is

$$\begin{aligned} P(S > R) &= P(S > R|R = 10)P(R = 10) + P(S > R|R = 13)P(R = 13) \\ &= [1 - F_S(10)] * 0.7 + [1 - F_S(13)] * 0.3 \\ &= \left[1 - \left(-\frac{10^3}{864} + \frac{10^2}{48}\right)\right] * 0.7 + [1 - 1] * 0.3 = 0.0519 \end{aligned}$$

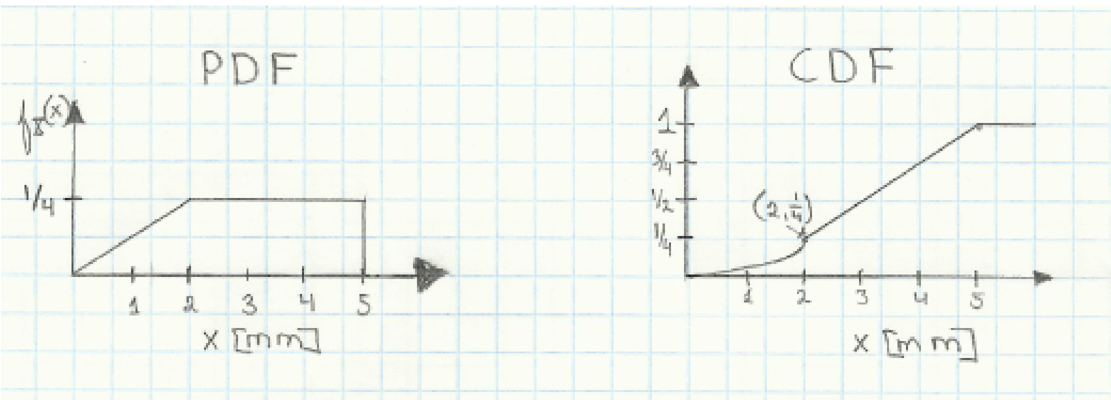
Problem 3.11

Let's define the event
$X =$ the size of a crack in a structural weld (in millimeter).

a) In order to sketch the CDF we need the CDF. Recall that $F_X(x) = \int_{-\infty}^{\infty} f_X(x)\, dx$ Therefore we integrate the PDF and get

$$F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{x^2}{16} & \text{for } 0 < x \le 2 \\ \frac{x}{4} - \frac{1}{4} & \text{for } 2 < x \le 5 \\ 1 & \text{for } x > 5 \end{cases}$$

sketches follow:



b) The mean crack size is

$$E(X) = \int_0^2 x\frac{x}{8}\, dx + \int_2^5 x\frac{1}{4}\, dx = \left[\frac{x^3}{24}\right]_0^2 + \left[\frac{x^2}{8}\right]_2^5 = \frac{71}{24} = 2.96(\text{mm})$$

c) The probability that a crack will be smaller than 4 mm is

$$P(X < 4) = 1 - P(X > 4)$$

Where $P(X > 4)$ is easily read off from the PDF, sketched in part a, as the area $(5-4)(1/4) = 1/4$, hence

$$P(X < 4) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

One could also find it the following way

$$F_X(4) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

d) To determine the median crack size we use the graph of the PDF, f_X. A vertical line drawn at the median x_m would divide the unit area under f_X into two equal halves; the right hand rectangle having area

$$0.5 = (5 - x_m)\left(\frac{1}{4}\right)$$

$$x_m = 5 - 4(0.5) = 3 \text{ (mm)}$$

Also using the graph of the CDF we get $F_X(x_m) = \frac{1}{2}$ which can easily found from the graph $x_m = 3$ (mm).

e) For part e we have a new event, lets define it.
Y = the number of cracks larger then 4 mm.
Each of the four cracks has $p = 0.25$ probability of exceeding 4 mm (as calculated in (c)). To determine the probability that only one of these four cracks is larger than 4 mm one uses binomial distribution. Where $n = 4$, and $p = 0.25$.

$$P(Y = 1) = \binom{4}{1} p^1 (1-p)^{(4-1)} = 4 * 0.25 * 0.75^3 = 0.422$$

Problem 3.40

Let's define the event

X = the settlement of a proposed structure (in inches).

The probability that the settlement of a proposed structure will not exceed 2 in. is

$$P(X \leq 2) = 0.95$$

The coefficient of variation of the settlement is $c.o.v. = 0.2$.
If a normal distribution is assumed for the settlement the probability that the proposed structure will settle more then 2.5 in. is

$$P(X > 2.5) = 1 - \Phi\left(\frac{x - \mu_X}{\sigma_X}\right)$$

where

$$\text{c.o.v.} = \frac{\sigma_X}{\mu_X} = 0.2 \quad \sigma_X = 0.2\mu_X$$

$$P(X \leq 2) = \Phi\left(\frac{2 - \mu_X}{0.2\mu_X}\right) = 0.95$$

or

$$\frac{2 - \mu_X}{0.2\mu_X} = \Phi^{-1}(0.95) = 1.645$$

$$\mu_X = 1.5$$

hence

$$P(X > 2.5) = 1 - \Phi\left(\frac{2.5 - 1.5}{0.2 * 1.5}\right) = 1 - \Phi(3.\bar{3}\bar{3}) = 0.000434$$

Problem 3.41

Let's define the event

$X =$ the strength of the concrete cylinder (in kips).

The strength of the cylinder is normally distributed as $N(80, 20)$ in kips therefore

$$\mu_X = 80 \quad \sigma_X = 20$$

a) To be a second place winner, X must be above 70 but below 100. The probability of winning second place is

$$\begin{aligned} P(70 < X < 100) &= \Phi\left(\frac{100-80}{20}\right) - \Phi\left(\frac{70-80}{20}\right) \\ &= \Phi(1) - \Phi(-0.5) = \Phi(1) - (1 - \Phi(0.5)) \\ &= 0.841 - 0.309 = 0.532 \end{aligned}$$

b) If the cylinder shows no sign of distress at a load of 90 kips the probability of winning first place is

$$\begin{aligned} P(X > 100|X > 90) &= \frac{P(X > 100 \text{ and } X > 90)}{P(X > 90)} = \frac{P(X > 100)}{P(X > 90)} \\ &= \frac{1 - \Phi\left(\frac{100-80}{20}\right)}{1 - \Phi\left(\frac{90-80}{20}\right)} = \frac{1 - \Phi(1)}{1 - \Phi(0.5)} = \frac{0.159}{0.309} = 0.514 \end{aligned}$$

For part c we need to define a new event

$Y =$ the strength of the **new** concrete cylinder (in kips).

$$\begin{aligned} \mu_Y &= 1.01 * 80 = 80.8 \\ \delta_Y &= 1.5 * \delta_X \end{aligned}$$

For $\mu_X > 0$ the c.o.v. is

$$\delta_X = \frac{\sigma_X}{\mu_X}$$

Therefore

$$\begin{aligned} \delta_Y &= 1.5 * \delta_X = 1.5 * \frac{\sigma_X}{\mu_X} = 1.5 * \frac{20}{80} = 0.375 \\ \sigma_Y &= \delta_Y * \mu_Y = 0.375 * 80.8 = 30.3 \end{aligned}$$

c) In order to determine which cylinder is likelier to score a higher strength we need the joint distribution of the event's. Lets define $Z = Y - X$. The probability of Z being larger the zero is

$$\begin{aligned} P(Z > 0) &= 1 - P(Z < 0) \\ &= \int_{-\infty}^{0}\int_{-\infty}^{0} f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{0}\int_{-\infty}^{0} f_X(x) f_Y(y)\, dx\, dy \\ &= \Phi\left(\frac{0-\mu_X}{\sigma_X}\right) * \Phi\left(\frac{0-\mu_Y}{\sigma_Y}\right) \\ &= \Phi(4) * \Phi(2.\bar{6}6) = 0.99606072 \end{aligned}$$