MAE 108: Solutions HW 3

Problem 1

Ang & Tang 2.39
We have
D : Number of defective panels on a given day
A : Shipment accepted on a given day

- $P(D = 0) = 0.2$
- $P(D = 1) = 0.5$
- $P(D = 2) = 0.3$

a)
The shipment is accepted if the supervisor finds at most one defected panel so $P(A) = P(D \leq 1)$ and $P(\bar{A}) = P(D = 2)$

$$\begin{aligned} P(A) &= P(D \leq 1) \\ &= P(D = 0) + P(D = 1) \\ &= 0.2 + 0.5 \\ &= 0.7 \end{aligned}$$

b)
P(exactly one shipment will be rejected in 5 days)

$$\begin{aligned} &= 5 * P(D \leq 1)^4 * P(D = 2) \\ &= 5 * 0.7^4 * 0.3 \\ &= 0.36 \end{aligned}$$

c)
The shipment is accepted if the supervisor finds at most one defected panel, however we now take into account that not all defected panel are found. Only 80% of the them are detected and therefore rejected, $P(\bar{A}|D = 1)$. Therefore our definition of the probability of acceptance of shipment on a given day $P(A)$ changes and is dependent on the number of defective panels **detected**, in stead of the number of defective panels.

D_d: the number of defective panels detected.

Therefore

- $P(\bar{A}|D=1) = 0.8$
- $P(A) = P(D_d \leq 1)$
- $P(\bar{A}) = P(D_d = 2)$

$$P(A) = P(A|D=0)P(D=0) + P(A|D=1)P(D=1) + P(A|D=2)P(D=2)$$

Keep in mind that A is now dependent on D_d when solving for the conditional probabilities and that $\bar{A}$: $D_d = 2$

$$P(A|D=0) = 1 - P(\bar{A}|D=0) = 1 - 0 = 1$$
$$P(A|D=1) = 1 - P(\bar{A}|D=1) = 1 - 0 = 1$$

This aligns with that there can not be a detection of two defected panels if there are not two defected panels to begin with. However when there are two defected panels we have:

$$\begin{aligned} P(A|D=2) &= 1 - P(\bar{A}|D=2) \\ &= 1 - P(\bar{A}|D=1)P(\bar{A}|D=1) \quad = 1 - 0.8 * 0.8 = 0.36 \end{aligned}$$

Hence, $P(A) = 0.2 + 0.5 + 0.36 * 0.3 = 0.808$
There are different ways of approaching this. Another approach is:

$$\begin{aligned} P(A) &= P(D_d \leq 1) \\ &= P(D_d \leq 1 | D \leq 1) * P(D \leq 1) + P(D_d \leq 1 | D = 2) * P(D = 2) \end{aligned}$$

By the implicit assumption, $D \leq 1$ implies $D_d \leq 1$.
Then, this means that $(D \leq 1) \subset (D_d \leq 1)$
and that

$$\begin{aligned} P(D_d \leq 1 | D \leq 1) &= P((D_d \leq 1) \cap (D \leq 1))/P(D \leq 1) \\ &= P(D \leq 1)/P(D \leq 1) = 1 \end{aligned}$$

$$P(D_d \leq 1|D = 2) = 1 - P(D_d = 2|D = 2) = 1 - (0.8)^2 = 0.36$$

Lets put this into the above equation

$$\begin{aligned}
P(A) &= P(D_d \leq 1) \\
&= P(D_d \leq 1|D \leq 1) * P(D \leq 1) + P(D_d \leq 1|D = 2) * P(D = 2) \\
&= 1 * (P(D = 0) + P(D = 1)) + [1 - P(D_d = 2|D = 2)]P(D = 2) \\
&= 0.2 + 0.5 + 0.36 * 0.3 = 0.808
\end{aligned}$$

Problem 2

Ang & Tang 2.45

Let A, D, and I denote the respective events that a driver encountering the amber light will accelerate, decelerate, or be indecisive. Let R denote the event that s/he will run the red light.
The given probabilities and conditional probabilities are:

- $P(A) = 0.10$
- $P(D) = 0.85$
- $P(I) = 0.05$
- $P(R|A) = 0.05$
- $P(R|D) = 0$
- $P(R|I) = 0.02$

a)
By the theorem of total probability,

$$\begin{aligned}
P(R) &= P(R|A)P(A) + P(R|D)P(D) + P(R|I)P(I) \\
&= 0.05 * 0.10 + 0 + 0.02 * 0.05 \\
&= 0.005 + 0.001 \\
&= 0.006
\end{aligned}$$

b) The desired probability is $P(A|R)$, which can be found by Bayes' Theorem as

$$\begin{aligned} P(A|R) &= \frac{P(R|A)P(A)}{P(R)} \\ &= \frac{0.05 * 0.10}{0.006} \\ &= 0.833 \end{aligned}$$

c) Let V mean there exists a vehicle waiting on the other street,

- $P(V) = 0.6$
- $P(\bar{V}) = 0.40$

Let C denote that the driver in the other vehicle is cautious,

- $P(C) = 0.8$
- $P(\bar{C}) = 0.20$

The probability of collision is:

$$P(\text{collision}) = P(\text{collision}|V)P(V) + P(\text{collision}|\bar{V})P(\bar{V})$$

$$\begin{aligned} P(\text{collision}|V) &= P(\text{collision}|C)P(C) + P(\text{collision}|\bar{C})P(\bar{C}) \\ &= (1 - 0.95) * 0.80 + (1 - 0.80) * 0.20 \\ &= 0.05 * 0.80 + 0.20 * 0.20 \\ &= 0.08 \\ P(\text{collision}|\bar{V}) &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} P(\text{collision}) &= P(\text{collision}|V)P(V) + P(\text{collision } |\bar{V})P(\bar{V}) \\ &= 0.08 * 0.60 + 0 \\ &= 0.048 \end{aligned}$$

d) 100,000 vehicles * 5% =5000 vehicles are expected to encounter the yellow light annually. Out of these 5000 vehicles, 0.6% (i.e. 0.006) are expected to run a red light, i.e. 5000*0.006 =30 vehicles. These 30 dangerous vehicles have 0.048 chance of getting into a collision (i.e. accident), hence 30* 0.048= **1.44** accidents caused by dangerous vehicles can be expected at the intersection per year

Problem 3

Ang & Tang 2.47

Let D denote difficult foundation problem, F denote that the project is in Ford County, I denote that the project in Iroquois County, and C denote a project in Champaign County.

$$\begin{aligned}
&P(D) = 2/3\\
&P(F) = 1/3 = 0.333\\
&P(I) = 2/5 = 0.4\\
&P(D|I) = 1.0\\
&P(D|F) = 0.5
\end{aligned}$$

a)

$$\begin{aligned}
P(F\bar{D}) &= P(\bar{D}|F)P(F)\\
&= 0.5 * 0.333 = 0.167
\end{aligned}$$

b)
We are asked to find the following:

$$P(C\bar{D}) = P(\bar{D}|C)P(C)$$

Therefore we need:

$$\begin{aligned}
P(\overline{D}|C) &= P(\overline{D}) = 1 - P(D) = 1/3\\
P(C) &= 1 - P(F) - P(I)\\
&= 1 - 0.333 - 0.4 = 0.267
\end{aligned}$$

Hence,

$$P(C\bar{D}) = P(\bar{D}|C)P(C) = 1/3 * 0.267 = 0.089$$

c)

$$P(I|\bar{D}) = \frac{P(\bar{D}|I)P(I)}{P(\bar{D})} = \frac{0 * 0.4}{0.667} = 0$$

Problem 4

Ang & Tang 2.51

Let C, S denote shortage of cement and steel bars respectively. The probabilities and condition probability are:

- $P(C) = 0.1$
- $P(S) = 0.05$
- $P(S|\bar{C}) = 0.5 * 0.05 = 0.025$

a)

$$P(S \cup C) = P(S) + P(C) - P(C|S)P(S)$$
$$P(\bar{C}|S) = \frac{P(S|\bar{C})P(\bar{C})}{P(S)} = \frac{0.025 * 0.9}{0.05} = 0.45$$
$$P(C|S) = 1 - P(\bar{C}|S) = 0.55$$
$$P(S \cup C) = 0.05 + 0.1 - 0.55 * 0.05 = 0.1225$$

b)

$$\begin{aligned} P(C\bar{S} \cup \bar{C}S) &= P(C\bar{S}) + P(\bar{C}S) \\ &= P(C \cup S) - P(CS) \text{ from Venn diagram} \\ &= 0.1225 - P(C|S)P(S) \\ &= 0.1225 - 0.55 * 0.05 \\ &= 0.095 \end{aligned}$$

c)

$$P(S|S \cup C) = \frac{P(S(S \cup C))}{P(S \cup C)} = \frac{P(S)}{P(S \cup C)} = \frac{0.05}{0.1225} = 0.408$$

For part d) and e) we have additional information.
Let U denote that the material was transported by tr**U**ck, A denote that the material is transported by tr**A**in, and T denote that the delivery was on time.

- $P(U) = 0.6$
- $P(A) = 0.4$
- $P(T|U) = 0.75$
- $P(T|A) = 0.9$

d)

$$\begin{aligned} P(T) &= P(T|U)P(U) + P(T|A)P(A) \\ &= 0.75 * 0.6 + 0.9 * 0.4 \\ &= 0.81 \end{aligned}$$

e)

$$\begin{aligned} P(U|\bar{T}) &= \frac{P(\bar{T}|U)P(U)}{P(\bar{T})} \\ &= \frac{[1 - P(T|U)] * P(U)}{[1 - P(T)]} \\ &= \frac{0.25 * 0.6}{[1 - 0.81]} \\ &= 0.7895 \end{aligned}$$

Problem 5

Ang & Tang 3.1
Total time $T = T_A + T_B$ its range is (3+4=7) to (5+6=11) Divide the

sample space into A=3, A=4 and A=5.

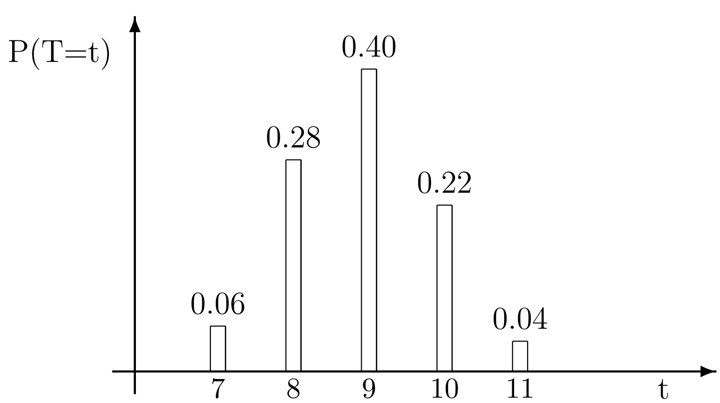
$$
\begin{aligned}
P(T=7) &= \sum_{n=3,4,5} P(T=7|A=n) * P(A=n) \\
&= \sum_{n=3,4,5} P(B=7-n) * P(A) \\
&= P(B=4) * P(A=3) \\
&= 0.2 * 0.3 = 0.06
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P(T=8) &= P(B=5) * P(A=3) + P(B=4) * P(A=4) \\
&= 0.6 * 0.3 + 0.2 * 0.5 = 0.28 \\
P(T=9) &= P(B=6) * P(A=3) + P(B=5) * P(A=4) + P(B=4) * P(A=5) \\
&= 0.2 * 0.3 + 0.6 * 0.5 + 0.2 * 0.2 = 0.4 \\
P(T=10) &= P(B=6) * P(A=4) + P(B=5) * P(A=5) \\
&= 0.2 * 0.5 + 0.6 * 0.2 = 0.22 \\
P(T=11) &= P(B=6) * P(A=5) \\
&= 0.2 * 0.2 = 0.04
\end{aligned}
$$

Lets check if it adds up:

$$0.06 + 0.28 + 0.4 + 0.22 + 0.04 = 1$$



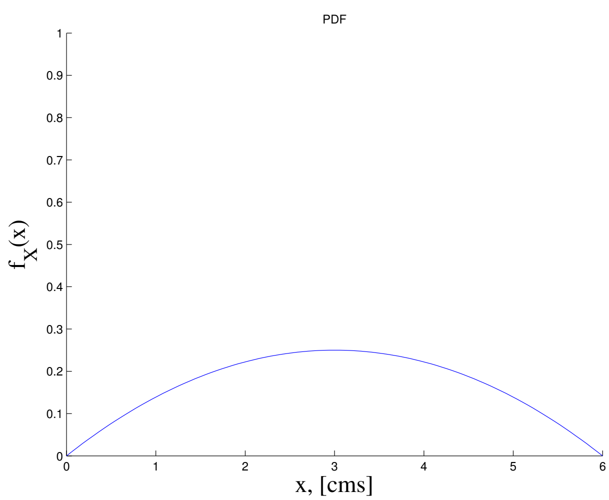
PMF

Problem 6

Ang & Tang 3.3

a) Applying the normalization condition we get:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$
$$\int_0^6 c\left(x - \frac{x^2}{6}\right)\,dx = 1$$
$$c\left[\frac{x^2}{2} - \frac{x^3}{18}\right]_0^6 = 1$$
$$c = \frac{18}{9 * 36 - 6^3} = 1/6$$



b) To avoid repeating integration , let's work with the Cumulative Distribution Function of X, which is

$$F_X(x) = \begin{cases} 0 & \text{for } s \leq 0 \\ \frac{1}{6}\left[\frac{x^2}{2} - \frac{x^3}{18}\right] = \frac{9x^2 - x^3}{108} & \text{for } 0 < x \leq 6 \\ 0 & \text{for } s > 12 \end{cases}$$

Since overflow already occurred, the given event is $X > 4$ (cm), hence the conditional probability

$$\begin{aligned} P(X < 5|X > 4) &= \frac{P(X < 5 \text{ and } X > 4)}{P(X > 4)} = \frac{P(4 < X < 5)}{1 - P(X \leq 4)} \\ &= \frac{F_X(5) - F_X(4)}{1 - F_X(4)} = \frac{(9 * 5^2 - 5^3) - (9 * 4^2 - 4^3)}{108 - (9 * 4^2 - 4^3)} \\ &= \frac{100 - 80}{108 - 80} = \frac{5}{7} = 0.714 \end{aligned}$$

c) Let C denote completion of pipe replacement by the next storm, where $P(C) = 0.06$. If C indeed occurs, overflow means $X > 5$, whereas if C did not occur then overflow would correspond to $X > 4$. Hence the total probability of overflow is

$$\begin{aligned} P(\text{overflow}) &= P(\text{overflow}|C)P(C) + P(\text{overflow}|\bar{C})P(\bar{C}) \\ &= P(X > 5) * 0.6 + P(X > 4) * (1 - 0.6) \\ &= [1 - F_X(5)] * 0.6 + [1 - F_X(4)] * 0.4 \\ &= (1 - 100/108) * 0.6 + (1 - 80/108) * 0.4 = 0.148 \end{aligned}$$

Problem 7

Ang & Tang 3.5

Let F be the final cost (a random variable), and C be the estimated cost (a constant), hence

$$X = F/C$$

is a random variable.
a) To satisfy the normalization condition,

$$\int_1^a \frac{3}{x^2}\,dx = \left[\frac{-3}{x}\right]_1^a = 3 - \frac{3}{a} = 1$$
$$a = 3/2 = 1.5$$

b) The given event asked for is F exceeds C by more then 25%. That can be written as:

$$F > 1.25 * C$$

or

$$F/C > 1.25$$

It follows that its probability is:

$$\begin{aligned}P(X > 1.25) &= \int_{1.25}^{\infty} f_X(x)\,dx \\ &= \int_{1.25}^{1.5} \frac{3}{x^2}\,dx = \left[\frac{-3}{x}\right]_{1.25}^{1.5} \\ &= -2 - (-2.4) = 0.4\end{aligned}$$

c) The mean

$$E(X) = \int_1^{1.5} x\frac{3}{x^2}\,dx = [3 \ln x]_1^{1.5} = 1.216$$

while

$$E(X^2) = \int_1^{1.5} x^2\frac{3}{x^2}\,dx = 3(1.5 - 1) = 1.5$$

with these, we can determine the variance

$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 1.5 - 1.216395324^2 = 0.020382415 \\ \sigma_X &= \sqrt{0.020382415} = 0.143\end{aligned}$$

Problem 8

Ang & Tang 3.7

a) The event roof failure in a given year means that the annual maximum snow load exceeds the design value, i.e. $X > 30$, whose probability is

$$\begin{aligned} P(X > 30) &= 1 - P(X \leq 30) = 1 - F_X(30) \\ &= 1 - [1 - (10/30)^4] \\ &= (1/3)^4 = 1/81 = 0.0123 = p \end{aligned}$$

Now for the first failure to occur in the 5th year, there must be four years of non-failure followed by one failure. We already found the value of failure,p, and therefore have the value of non-failure,$1 - p$. The probability of such an event is:

$$(1-p)^4 p = [1 - (1/81)]^4 * (1/81) = 0.0117$$

b) Among the next 10 years , let Y count the number of years in which failure occurs. Y follows a binomial distribution with n= 10 and p =1/81, hence

$$\begin{aligned} P(Y < 2) &= P(Y = 0) + P(Y = 1) \\ &= (1-p)^n + n(1-p)^{n-1} p \\ &= (80/81)^{10} + 10 * (80/81)^9 * (1/81) \\ &= 0.994 \end{aligned}$$