# MAE108 S2014 - Homework 2 Solutions

## Problem 2.4

Let's define $\Omega_1 = \{6, 7, 8\}$ ft/day as the set of possible inflow rates and $\Omega_2 = \{5, 6, 7\}$ ft/day as the set of possible outflow rates.

a) The possible combinations (inflow,outflow) of water in a given day are the following pairs: $(6, 5), (6, 6), (6, 7), (7, 5), (7, 6), (7, 7), (8, 5), (8, 6), (8, 7)$ ft/day.

b) The possible changes in water level is given by the set $C = \{y - x \,|\, y \in \Omega_1, x \in \Omega_2\}$, so $C = \{-1, 0, 1, 2, 3\}$ ft/day. If the water level in the tank is 7 ft from the bottom at the start of a day, then at the end of the day there can be 6, 7, 8, 9, or 10 ft of water in the tank.

c) There are 3 combinations of inflow and outflow that result in the tank having at least 9 ft of water at the end of the day, these combinations are $(7, 5), (8, 6), (8, 5)$ ft/day. If the amounts of inflow and outflow of water are both equally likely and there are 9 total combinations, then the probability of this happening is $\frac{3}{9} = \frac{1}{3}$.

## Problem 2.6

We will define the following events:

$A_1$ = lane $A_1$ requires major resurfacing in the next 2 years.
$A_2$ = lane $A_2$ requires major resurfacing in the next 2 years.
$B_1$ = lane $B_1$ requires major resurfacing in the next 2 years.
$B_2$ = lane $B_2$ requires major resurfacing in the next 2 years.

We know $P(A_1) = 0.05$, $P(A_2) = 0.05$, $P(B_1) = 0.15$, $P(B_2) = 0.15$, $P(A_2|A_1) = 0.15$, $P(A_1|A_2) = 0.15$, $P(B_2|B_1) = 0.45$, and $P(B_1|B_2) = 0.45$.

a) The probability that route $A$ will require major resurfacing in the next two years is given by

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 P_2) \\ &= P(A_1) + P(A_2) - P(A_1|A_2)P(A_2) \\ &= 0.05 + 0.05 - 0.15 * 0.05 \\ &= 0.0925. \end{aligned}$$

The probability that route $B$ will require major resurfacing in the next two years is given by

$$\begin{aligned} P(B_1 \cup B_2) &= P(B_1) + P(B_2) - P(B_1 B_2) \\ &= P(B_1) + P(B_2) - P(B_1|B_2)P(B_2) \\ &= 0.15 + 0.15 - 0.45 * 0.15 \\ &= 0.2325. \end{aligned}$$

b) Assuming that the need for resurfacing in routes $A$ and $B$ are indepedent, the probability that the road between cities 1 and 3 needs resurfacing is given by

$$\begin{aligned} P(A_1 \cup A_2 \cup B_1 \cup B_2) &= P(A_1 \cup A_2) + P(B_1 \cup B_2) - P([A_1 \cup A_2] \cap [B_1 \cup B_2]) \\ &= P(A_1 \cup A_2) + P(B_1 \cup B_2) - P(A_1 \cup A_2)P(B_1 \cup B_2) \\ &= 0.0925 + 0.2325 - 0.0925 * 0.2325 \\ &= 0.304. \end{aligned}$$

## Problem 2.8

We will define the following events:

$E_1$ = there will be no rain.

$E_2$ = production of concrete material at the job site is feasible.

$E_3$ = supply of premixed concrete is available.

We know $P(E_1) = 0.8$, $P(E_2) = 0.7$, $P(E_3) = 0.95$, $P(E_3|\overline{E}_2) = 0.6$, and $E_1$ is statistically independent of $E_2$ and $E_3$.

a) We know $A$ = casting of concrete elements can be performed on a given day. We can write $A = E_1(E_2 \cup E_3)$, because we require no rain, and either on-site production or premixed concrete (or both).

$B = \overline{E_1(E_2 \cup E_3)} = \overline{E}_1 \cup (\overline{E}_2\overline{E}_3)$.

b) The probability of event $B$ is given by

$$
\begin{aligned}
P(B) &= P(\overline{E}_1 \cup (\overline{E}_2\overline{E}_3)) \\
&= P(\overline{E}_1) + P(\overline{E}_2\overline{E}_3) - P(\overline{E}_1\overline{E}_2\overline{E}_3) \\
&= (1 - P(E_1)) + P(\overline{E}_3|\overline{E}_2)P(\overline{E}_2) - P(\overline{E}_1)P(\overline{E}_2\overline{E}_3) \\
&= (1 - P(E_1)) + (1 - P(E_3|\overline{E}_2))(1 - P(E_2)) - (1 - P(E_1))(1 - P(E_3|\overline{E}_2))(1 - P(E_2)) \\
&= (1 - 0.8) + (1 - 0.6)(1 - 0.7) - (1 - 0.8)(1 - 0.6)(1 - 0.7) \\
&= 0.296.
\end{aligned}
$$

c) The probability that casting concrete can be performed given production of concrete is not feasible is

$$
\begin{aligned}
P(A|\overline{E}_2) &= P(E_1(E_2 \cup E_3)|\overline{E}_2) \\
&= \frac{P(E_1(E_2 \cup E_3)\overline{E}_2)}{P(\overline{E}_2)} \\
&= \frac{P(E_1\overline{E}_2E_3)}{P(\overline{E}_2)} = \frac{P(E_1)P(E_3|\overline{E}_2)P(\overline{E}_2)}{P(\overline{E}_2)} \\
&= 0.8 * 0.6 = 0.48.
\end{aligned}
$$

## Problem 2.10

We will define the following events:

$A$ = contractor $A$ is available.

$B$ = contractor $B$ is available.

We know $P(A) = 0.6$, $P(B) = 0.8$, and $P(A \cup B) = 0.9$.

a) The probability that both subcontractors will be available is $P(AB)$, which can be obtained from

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(AB) \\
P(AB) &= P(A) + P(B) - P(A \cup B) \\
&= 0.6 + 0.8 - 0.9 = 0.5.
\end{aligned}
$$

b) The probability that $B$ is available given $A$ is not available is given by

$$\begin{aligned} P(B|\overline{A}) &= \frac{P(B)}{P(\overline{A})} P(\overline{A}|B) \\ &= \frac{P(B)}{1-P(A)} (1 - P(A|B)) \\ &= \frac{P(B)}{1-P(A)} \left(1 - \frac{P(AB)}{P(B)}\right) \\ &= \frac{0.8}{1-0.6} \left(1 - \frac{0.5}{0.8}\right) = 0.75. \end{aligned}$$

c) i) For $A$ and $B$ to be statistically independent, $P(A|B) = P(A)$ and $P(B|A) = P(B)$. However, we can see that $P(B|A) = 0.75$ while $P(B) = 0.8$.

ii) For $A$ and $B$ to be mutually exclusive, $P(AB) = 0$. Because $P(AB) = 0.5$, $A$ and $B$ are not mutually exclusive.

iii) For $A$ and $B$ to be collectively exhaustive, $P(A \cup B) = 1$. Because $P(A \cup B) = 0.9$, $A$ and $B$ are not collectively exhaustive.

## Problem 2.12

We will define the following events:
$A$ = flooding of town $A$ in a given year.
$B$ = flooding of town $B$ in a given year.
$C$ = flooding of town $C$ in a given year.
We know $P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.1$, $P(B|C) = 0.6$, $P(A|BC) = 0.8$, and $P(\overline{A}\,\overline{B}|\overline{C}) = 0.9$.

a) The probability that a given year is a disaster is given by

$$\begin{aligned} P(ABC) &= P(A|BC)P(BC) \\ &= P(A|BC)P(B|C)P(C) = 0.8 * 0.6 * 0.1 = 0.048. \end{aligned}$$

b) The probability that $C$ floods given $B$ floods is given by

$$\begin{aligned} P(C|B) &= \frac{P(C)}{P(B)} (P(B|C)) \\ &= \frac{0.1}{0.3} (0.6) = 0.2. \end{aligned}$$

c) The probability that at least one town is flooded is given by

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P(\overline{A \cup B \cup C}) \\ &= 1 - P(\overline{A}\,\overline{B}\,\overline{C}) = 1 - P(\overline{A}\,\overline{B}|\overline{C})P(\overline{C}) \\ &= 1 - 0.9 * (1 - 0.1) = 0.19. \end{aligned}$$

## Problem 2.14

a) We will define $A$ as an event where an accident will occur at a given crossing next year. Assuming the likelihood of a crash at each crossing is the same and given an average of 13 accidents per year, its probability is

$$\begin{aligned} P(A) &= 1 - P(\overline{A}) \\ &= 1 - \left(\frac{999}{1000}\right)^{13} = 0.0129. \end{aligned}$$

Note that this is very close to $0.0013 = \frac{13}{1000}$, because there is a very small chance that a given crossing will have more than one accident.

b) The probability of a struck by train accident occuring during daytime is

$$P(S|D) = \frac{60}{90} = \frac{2}{3}.$$

c) We will denote $F$ as a fatal accident.

$$\begin{aligned} P(F) &= P(F|R)P(R) + P(F|S)P(S) \\ &= 0.5 * \frac{50}{130} + 0.8 * \frac{80}{130} = 0.685. \end{aligned}$$

d) i) $D$ and $R$ are not mutually exclusive, because $P(DR) > 0$.

ii) $D$ and $R$ are not statistically independent, because $P(R|D) = \frac{30}{90} \neq P(R)$, which is $\frac{50}{130}$.

## Problem 2.16

We will define the following events:
$E_1$ = Monday is a rainy day.
$E_2$ = Tuesday is a rainy day.
$E_3$ = Wednesday is a rainy day.

a) We know

1) $P(E_1) = P(E_2) = P(E_3) = 0.3$.
2) $P(E_2|E_1) = P(E_3|E_2) = 0.5$.
3) $P(E_3|E_1E_2) = 0.2$.

b) The probability of rain on Monday and Tuesday is

$$\begin{aligned} P(E_1E_2) &= P(E_2|E_1)P(E_1) \\ &= 0.5 * 0.3 = 0.15. \end{aligned}$$

c) The probability that Wednesday is the only dry day of the three days is

$$\begin{aligned} P(E_1E_2\overline{E}_3) &= P(\overline{E}_3|E_1E_2)P(E_1E_2) \\ &= (1 - P(E_3|E_1E2))P(E_2|E_1)P(E_1) = (1 - 0.2)(0.5)(0.3) = 0.12. \end{aligned}$$

d) The probability of at least one rainy day is

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_1E_3) - P(E_2E_3) + P(E_1E_2E_3) \\ &= 0.3 + 0.3 + 0.3 - 0.15 - 0.15 - 0.075 + 0.03 = 0.56 \end{aligned}$$

$$\begin{aligned} P(E_1E_3) &= P(E_3|E_1)P(E_1) = (P(E_3|E_2E_1)P(E_2|E_1) + P(E_3|\overline{E}_2E_1)P(\overline{E}_2|E_1))P(E_1) \\ &= (0.2 * 0.5 + 0.3 * 0.5)(0.3) = 0.075 \\ P(E_2E_3) &= P(E_3|E_2)P(E_2) = 0.5 * 0.3 = 0.15 \\ P(E_1E_2E_3) &= P(E_3|E_1E_2)P(E_1E_2) = 0.2 * 0.15 = 0.03 \end{aligned}$$

**Problem 2.18**

We will define the following events:

$E$ = excessive settlement over a building's lifetime

$C$ = Collapse of the superstructure over a building's lifetime.

We know $P(E) = 0.10$, $P(C) = 0.05$, and $P(C|E) = 0.20$.

a) Probability of building failure is

$$\begin{aligned} P(E \cup C) &= P(E) + P(C) - P(CE) \\ &= P(E) + P(C) - P(C|E)P(E) = 0.10 + 0.05 - 0.20 * 0.10 = 0.13. \end{aligned}$$

b) The probability of both failure modes given some type of failure is

$$\begin{aligned} P(EC|E \cup C) &= \frac{P(EC) \cap (E \cup C)}{P(E \cup C)} \\ &= \frac{P(C|E)P(E)}{P(E \cup C)} = \frac{0.20 * 0.10}{0.13} = 0.154. \end{aligned}$$

c) The probability that only one mode will fail over the life of the building is

$$\begin{aligned} P(E\overline{C} \cup \overline{E}C) &= P(\overline{C}|E)P(E) + P(\overline{E}|C)P(C) \\ &= (1 - P(C|E))P(E) + (1 - P(E|C))P(C) \\ &= (1 - P(C|E))P(E) + (1 - \frac{P(E)}{P(C)}P(C|E))P(C) \\ &= (1 - 0.20)0.10 + (1 - \frac{0.10}{0.05}0.20)0.05 = 0.11. \end{aligned}$$