MAE 108 – Probability and Statistical Methods for Engineers - Spring 2013

Midterm # 2 - May 24, 2012

50 minutes, open book, open notes, calculator allowed, no cell phones or computers.

Write your answers directly on the exam. 40 points total.

The exam is **5** pages long. **Justify all your answers**.

NAME:

EXAM GRADE:

(1) (10 points) A company receives 60% of its orders over the internet. The orders are statistically independent from each other. Within a collection of 6 independently placed orders, what is:

(a) The probability that between 2 and 4 of the orders are received over the internet?

(3 points) Solution: We clearly have a Bernoulli sequence with $p = 0.6$ and $n = 6$. In this question we are asked to compute $P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{6!}{2!4!}0.6^2 0.4^4 + \frac{6!}{3!3!}0.6^3 0.4^3 + \frac{6!}{4!2!}0.6^4 0.4^2 = 0.6^2 0.4^2[15 \times 0.4^2 + 20 \times 0.6 \times 0.4 + 15 \times 0.6^2] = 72.6\%$

(b) The probability that at most 4 of the orders are received over the internet?

(2 points) Solution: $P(X \leq 4) = 1 - P(X = 5) - P(X = 6) = 1 - 6 \times 0.6^5 \times 0.4 - 0.6^6 = 76.7\%$

(c) The probability that the fifth order received is the first internet order?

(3 points) Solution: This is the probability that the first occurrence of an internet order is at the fifth order, and thus $P = 0.4^4 \times 0.6 = 1.53\%$

(d) The probability that there are no internet orders in the last 3 orders given that we know there are no internet orders in the first 3.

(2 points) Solution: The trials in a Bernoulli sequence are statistically independent so what happens in the first 3 orders is irrelevant to the last three. The probability is thus the same as the probability to have no internet orders in the last three orders, which is $P = 0.4^3 = 6.4\%$.

(2) (10 points) The arrival times of patients in an emergency room (ER) follow a Poisson process with an average of 1.8 new patients per hour.

(a) What is the expected value of the time between two arrivals in the ER?

(3 points) Solution: We have a Poisson process with $\nu = 1.8$. Here we are asked about the recurrence time, T. So $E(T) = \frac{1}{\nu} = 0.55$ hour ≈ 33 minutes.

(b) What is the probability that at least 2 patients arrive in the ER within a 2-hour period?

(4 points) Solution: Let us call X the number of patients. We want to compute $P(X_2 \geq 2) = 1 - P(0) - P(1)$. For each of these we have to apply the Poisson formula with $\nu t = 1.8 \times 2 = 3.6$ and we find $P(X_2 \geq 2) = 1 - e^{-3.6} - 3.6e^{-3.6} \approx 87.4\%$.

(c) What is the probability that there is more than one hour between two arrivals in the ER?

(3 points) Solution: This is a question regarding the recurrence time. We are asked to compute $P(T > 1)$. This is the same as the probability that there are no arrivals within a one-hour window, and thus $P(T > 1) = P(X_1 = 0) = e^{-1.8} \approx 16.5\%$.

(3) (10 points)

A continuous random variable, X, has a **constant** probability distribution function (pdf) on the interval $[a, b]$ (and the pdf is zero outside that interval).

(a) Determine the standard deviation, σ, for this random variable as a function of a and b.

[Hint: the problem statement allows you to determine the value for the pdf.]

(6 points) Solution: We need to determine the mean value of X and its variance. Since the pdf is constant and it has to have integral 1, we necessarily have $f_X = \frac{1}{b-a}$. To get the mean we have to compute

$$E(X) = \int_a^b \frac{x}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

To compute the variance we first need to calculate the mean of X^2 and we have

$$E(X^2) = \int_a^b \frac{x^2}{b-a} dx = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

therefore the variance is given by

$$Var(X) = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)}{12} = \frac{(b-a)^2}{12}$$

and the result for the standard deviation is

$$\sigma = \sqrt{Var(x)} = \frac{b-a}{2\sqrt{3}}$$

(b) What is the probability for the random variable to be within exactly one standard deviation from the mean?

(4 points) Solution: Since the pdf is constant the probability is simply the size of the interval over the size of the sample space. The set of values of X which are within 1 standard deviation from the mean is of size 2σ. The size of the sample space is $b - a$ therefore the probability is $1/\sqrt{3} \approx 57.7\%$

(4) (10 points) [Harder] You are investing your money in mutual funds and have decided to split the money among 2 funds, A and B. If $x is invested in fund A, its worth after one year is a normal distribution with mean $\mu_A = 1.05x$ and variance $\sigma_A = 0.02x^2$. If instead $x is invested in fund B, its net worth after one year is a normal distribution with mean $\mu_B = 1.05x$ and variance $\sigma_B = 0.03x^2$. Suppose you are interested in investing a total of $1000, which you are splitting with $y invested in fund A and $(1000-y) invested in funds B. Assume that both funds are statistically independent. [Hint: you will need to recall the results seen in Chapter 4 about the sum of two normal distributions]

(a) What is the expected value of the total worth of your investment after one year?

(2 points) Solution: Let us denote by M your total amount of money, X_A the normal variable from fund A and X_B that from fund B. We have $M = X_A + X_B$ and thus $E(M) = E(X_A) + E(X_B) = 1.05y + 1.05(1000 - y) = \$1,050$.

(b) What is the variance of the total worth of your investment after one year? (no need to expand your answer)

(2 points) Solution: Since $M = X_A + X_B$ we can apply the formula for the variance of the sum of 2 independent variables, and find $Var(M) = Var(X_A) + Var(X_B) = 0.02y^2 + 0.03(1000 - y)^2$.

(c) What is the value of y which minimizes the variance of the total worth of your investment after one year?

(3 points) Solution: To find the minimum of $Var(M)$ we take its derivative with respect to y and find that the derivative is $d[Var(M)]/dy = 2[0.02y - 0.03(1000 - y)]$ which is zero when $y = 600$.

(d) If you adopt the value of y determined in question (c), what is the probability that after one year the total worth of your investment is more than $1,110?

[Hint: you will need to use the table in Appendix A]

(3 points) Solution: If we take $y = 600$ then the variance of M is 12000, and thus $M = N(1050, \sqrt{12000}) = N(1050, 109.5)$ (M is the sum of 2 normal variables and is thus normally distributed). We want to compute $P(M \geq 1110) = 1 - \Phi(\frac{1110-1050}{109.5}) = 1 - \phi(0.55) \approx 29\%$.