NAME:

MAE 108 – Probability and Statistical Methods for Engineers - Spring 2012

Midterm # 1 - May 4, 2012

50 minutes, open book, open notes, calculator allowed, no cell phones or computers.

Write your answers directly on the exam. 50 points total.

The exam is **5** pages long. **Justify all your answers**.

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(1) (10 points) Consider the roll of a fair dice. A high score is a 4, 5 or a 6. An even score is a 2, 4, or 6.

(a) Are the events "even" and "high score" statistically independent?

Solution: Let us say E is even and H high score. Clearly $P(E) = 1/2$ and $P(H) = 1/2$. The event EH is simply: $\{4, 6\}$ hence $P(EH) = 1/3 \neq P(E) \times P(H)$ hence the events are not independent.

(b) If the face number is even, what is the probability that it is a high score?

Solution: $P(H|E) = P(EH)/P(E) = 2/3$.

(c) What is the probability of rolling neither an even number nor a high score?

Solution: We have $\bar{E}\bar{H} = \{1, 3\}$ hence $P(\bar{E}\bar{H}) = 1/3$.

(2) (10 points) A class of 100 students has two sections. Section I has 55 students of whom 10 received A grades. Section II has 45 students of whom 11 received A grades. A student is chosen at random in the class.

(a) What is the probability that the student received a grade A?

Solution: Let us denote 'A' the event: the student received a grade of A. Here we want to know $P(A)$. We use the theorem of total probability. The two mutually exclusive and collectively exhaustive events are S1 and S2. So we have $P(A) = P(A|S1)P(S1) + P(A|S2)P(S2) = \left(\frac{10}{55} \times \frac{55}{100}\right) + \left(\frac{11}{45} \times \frac{45}{100}\right) = 21\%$

(b) If the student received a grade A, what is the probability he was in section I?

Solution: Here we have to calculate $P(S1|A)$ and use Bayes's theorem $P(S1|A) = P(S1A)/P(A) = P(A|S1)P(S1)/P(A) = \left(\frac{10}{55}\right) \times \frac{55/100}{21/100} = \frac{10}{21} \approx 47.6\%$.

(3) (10 points) A continuous random variable takes values between 0 and 4 with a cumulative distribution function given by

$$F_X(x) = \frac{x^2}{32} + Ax + B,$$

where A and B are two unknown constants.

(a) Using the two boundary conditions for F_X determine the values of A and B.

Solution: $F_X(0) = 0$ hence $B = 0$. We also have $F_X(4) = 1$ hence $A = 1/8$.

(b) What is the probability distribution function for this random variable?

Solution: The PDF f_X is equal to dF_X/dx so $f_X(x) = x/16 + 1/8$.

(c) Calculate the mean value of the random variable, $E(X)$.

Solution: The mean is given by the integral

$$E(X) = \int_0^4 x f_X(x)dx = \int_0^4 (x^2/16 + x/8)dx = [x^3/48 + x^2/16]_0^4 = 64/48 + 1 = 7/3.$$

(d) If you know that X is less than 2, what is the probability that it is less than 1?

Solution: $P(X \leq 1 | X \leq 2) = \frac{P(X \leq 1 \cap X \leq 2)}{P(X \leq 2)} = \frac{P(X \leq 1)}{P(X \leq 2)} = \frac{F_X(1)}{F_X(2)} = \frac{1/32+1/8}{4/32+2/8} = \frac{5}{12} \approx 41.6\%$.

(4) (10 points) A construction company employs 2 sales engineers. Engineer 1 does the work in estimating cost for 70% of jobs bid by the company while engineer 2 does the work for 30% of those jobs. It is known that the error rate for engineer 1 is such that 2% is the probability of an error when he does the work, whereas the probability of an error in the work of engineer 2 is 4%. A bid arrives and a serious error occurs in estimating cost. Which engineer would you guess did the work? Justify your answer.

Solution: Let us call 1 and 2 the number of the engineers and E the "error" event. We have $P(1) = 0.7$ and $P(2) = 0.3$. We are also given that $P(E|1) = 2\%$ and $P(E|2) = 4\%$. We thus can calculate $P(1|E) = P(E|1)P(1)/P(E)$. To get $P(E)$ we apply the theorem of total probability: $P(E) = P(E|1)P(1) + P(E|2)P(2) = 2.6\%$ and thus $P(1|E) = 53.85\%$. Similarly you can get $P(2|E) = 46.15\%$ hence it is more likely that engineer 1 did the error.

(5) (10 points) Assume you own a fair coin and throw it four times. Call X the random variable denoting the difference between the number of heads and the number of tails (X could therefore be positive or negative). Determine the probability mass function (PMF) for X.

Solution: First what are the possible values for X?: it could be 4 (4 heads, 0 tails), 2 (3 heads, 1 tails), 0 (2 heads and 2 tails), -2 (1 heads, 3 tails) and -4 (0 heads, 4 tails). The sample space is of size 16 since we have 2 choices at each throw and have thus a size $2 \times 2 \times 2 \times 2 = 16$. The values 4 and -4 can only realized in one way: TTTT or HHHH, thus have each a PMF equal to 1/16. The values 2 and -2 can each be realized in 4 different ways (for 2 they are: HHHT, HHTH, HTHH, THHH and for -2 they are TTTH, TTHT, THTT, HTTT), corresponding to a PMF of 1/4. The case $X = 0$ can then be deduced as the complementary of the other ones, and we thus have the PMFs: $p_X(4) = 1/16$, $p_X(2) = 1/4$, $p_X(0) = 3/8$, $p_X(-2) = 1/4$ $p_X(-4) = 1/16$.